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PROJECTION OF AN INVARIANT LOCUS IN [8] FROM A SOLID LYING ON IT

By A. F. HORADAM (*Armidale*)

[Received 23 August 1956; in revised form 12 November 1957]

1. Introduction

Previously in (3), I have shown that in [8] there is a locus L, of order 45 and dimension 4 which is invariant under the 51840×81 operations of a certain group. This locus L is determined by the 5 independent cubic primals

$$\theta \equiv \sum_{i,j=0}^{2} x_{ij}^3 = 0, \tag{i}$$

$$\phi_1 \equiv x_{00}x_{01}x_{02}+x_{10}x_{11}x_{12}+x_{20}x_{21}x_{22} = 0, \tag{ii}$$

$$\phi_2 \equiv x_{00}x_{10}x_{20}+x_{01}x_{11}x_{21}+x_{02}x_{12}x_{22} = 0, \tag{iii}$$

$$\phi_3 \equiv x_{00}x_{11}x_{22}+x_{01}x_{12}x_{20}+x_{02}x_{10}x_{21} = 0, \tag{iv}$$

$$\phi_4 \equiv x_{00}x_{12}x_{21}+x_{01}x_{10}x_{22}+x_{02}x_{11}x_{20} = 0, \tag{v}$$

where $(x_{00}, x_{10}, x_{20}, x_{01}, x_{11}, x_{21}, x_{02}, x_{12}, x_{22})$ is the ordered set of homogeneous coordinates in [8]. Let $A_{00}, \ldots, A_{22}$ be the vertices of the simplex of reference in [8].

On L there lie 81 solids with which are associated 81 [4]'s not lying on L. Each pair of such spaces, solid and [4], is invariant under the collineation of an involutory matrix of the group. One such pair concerns us, namely the solid j determined by the points

$$A_{10}-A_{20}, \quad A_{01}-A_{02}, \quad A_{11}-A_{22}, \quad A_{21}-A_{12}$$

or by the primes

$$x_{00} = 0, \quad x_{10}+x_{20} = 0, \quad x_{01}+x_{02} = 0, \quad x_{11}+x_{22} = 0, \quad x_{21}+x_{12} = 0,$$

and the [4] J determined by the primes

$$x_{10}-x_{20} = 0, \quad x_{01}-x_{02} = 0, \quad x_{11}-x_{22} = 0, \quad x_{21}-x_{12} = 0.$$

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2. Projection of L from a solid lying on it

We now proceed to project L from j on to J. Firstly, make the transformation whose matrix is

$$K = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & . & . & . & . & . & . & . & . \\ . & 1 & 1 & . & . & . & . & . & . \\ . & 1 & -1 & . & . & . & . & . & . \\ . & . & . & 1 & . & . & 1 & . & . \\ . & . & . & . & 1 & . & . & . & 1 \\ . & . & . & . & . & 1 & . & 1 & . \\ . & . & . & 1 & . & . & -1 & . & . \\ . & . & . & . & . & 1 & . & -1 & . \\ . & . & . & . & 1 & . & . & . & -1 \end{bmatrix},$$

so that $K^2 = I$.

The solid j now has the transformed equations (omitting dashes),

$$x_{00} = x_{10} = x_{01} = x_{11} = x_{21} = 0,$$

and the original defining points of the solid are transformed into the points A_{20}, A_{02}, A_{22}, A_{12}.

Further, the 5 independent cubic loci defining L have the transformed equations

$$\theta \equiv x_{00}^3+2x_{10}^3+6x_{10}x_{20}^2+2x_{01}^3+6x_{01}x_{02}^2+2x_{11}^3+6x_{11}x_{22}^2+ \\ +2x_{21}^3+6x_{21}x_{12}^2 = 0, \quad \text{(i')}$$

$$\phi_1 \equiv x_{00}(x_{01}^2-x_{02}^2)+2x_{10}(x_{11}x_{21}-x_{12}x_{22})+2x_{20}(x_{21}x_{22}-x_{11}x_{12}) = 0, \quad \text{(ii')}$$

$$\phi_2 \equiv x_{00}(x_{10}^2-x_{20}^2)+2x_{01}(x_{11}x_{21}+x_{12}x_{22})+2x_{02}(x_{11}x_{12}+x_{21}x_{22}) = 0, \quad \text{(iii')}$$

$$\phi_3 \equiv x_{00}(x_{11}^2-x_{22}^2)+2x_{10}(x_{01}x_{21}-x_{02}x_{12})+2x_{20}(x_{01}x_{12}-x_{21}x_{02}) = 0, \quad \text{(iv')}$$

$$\phi_4 \equiv x_{00}(x_{21}^2-x_{12}^2)+2x_{10}(x_{01}x_{11}-x_{02}x_{22})+2x_{20}(x_{11}x_{02}-x_{01}x_{22}) = 0, \quad \text{(v')}$$

when again dashes are omitted.

The section of this cubic by J, of which the equations are now $x_{20} = x_{02} = x_{12} = x_{22} = 0$, is the set of 5 equations, given by Horadam (3), determining the Burkhardt configuration in [4].

To obtain the projection of L from j, we eliminate x_{20}, x_{02}, x_{12}, x_{22} from equations (i'),..., (v'). This is done by multiplying (i') by x_{00} and in this equation substituting in turn for $x_{00}x_{02}^2$, $x_{00}x_{20}^2$, $x_{00}x_{22}^2$, $x_{00}x_{12}^2$, which are obtained from (ii'), (iii'), (iv'), (v') respectively on multiplication by x_{02}, x_{20}, x_{22}, x_{12} respectively. Accordingly, after simplification, we get the quartic primal

$$x_{00}^4+8x_{00}(x_{10}^3+x_{01}^3+x_{11}^3+x_{21}^3)+48x_{10}x_{01}x_{11}x_{21} = 0.$$

If we replace x_{00} by $-2x_{00}$, while leaving the remaining variables unaltered, we find that the equation of this quartic locus reduces to the form

$$x_{00}^4 - x_{00}(x_{10}^3+x_{01}^3+x_{11}^3+x_{21}^3)+3x_{10}x_{01}x_{11}x_{21} = 0, \qquad \text{(vi)}$$

which is precisely the equation of B_3^4, the rational *Burkhardt primal* in [4], given by Baker (1). Thus, we have

THEOREM 1. *The projection of L from a solid lying on it onto a* [4] *is the Burkhardt primal.*

This is quite a remarkable result. Since the fourfold locus L_4^{45} projects from the solid j into a threefold locus B_3^4, every point on B is the projection, not of a point, but of a curve on L. The nature of this curve is investigated next.

3. A net of quadric surfaces in [3]

Through the solid j take a system of [4]'s. Then the projections of these in J yield points of the projected locus B. A general [4] of the system is given by

$$\frac{x_{00}}{1} = \frac{x_{10}+x_{20}}{a} = \frac{x_{11}+x_{22}}{b} = \frac{x_{12}+x_{21}}{c} = \frac{x_{01}+x_{02}}{d}, \qquad \text{(vii)}$$

where a, b, c, d are parameters.

In order to obtain the projection, we eliminate $x_{20}, x_{21}, x_{22}, x_{02}$ (say) from equations (i),..., (v), using (vii), where a, b, c, d are allowed to vary. Performing this elimination and, for convenience in manipulation, writing x, y, z, w, t for $x_{10}, x_{11}, x_{12}, x_{00}, x_{01}$ respectively, we find, on removing the factor w which occurs in each equation, that

$$\theta \equiv (1+a^3+b^3+c^3+d^3)w^3 - 3w(d^2t+a^2x+b^2y+c^2z)+ \\ +3(dt^2+ax^2+by^2+cz^2) = 0,$$

$$\phi_1 \equiv t^2 - ayz - bzx - cxy + w(-dt+bcx+cay+abz) - abcw^2 = 0,$$

$$\phi_2 \equiv x^2 + dyz - w(ax+bdz) - t(cy-bz) = 0,$$

$$\phi_3 \equiv y^2 + dzx - w(by+cdx) - t(az-cx) = 0,$$

$$\phi_4 \equiv z^2 + dxy - w(cz+day) - t(bx-ay) = 0.$$

It must be emphasized that these are not transforms of the original θ, ϕ_i since the factor w has been omitted throughout. The preservation of the original symbolism is done for convenience. Hereafter, we refer to these loci and not to the original θ, ϕ_i defining L.

We may regard θ, ϕ_i as the equations of quadrics in the [4]. Their

intersection is the locus, or set of points, in which L is cut by the [4] through j. Now

$$\theta-3(d\phi_1+a\phi_2+b\phi_3+c\phi_4) = w^2(1+a^3+b^3+c^3+d^3+3abcd) = 0.$$

Write $\Delta = 1+a^3+b^3+c^3+d^3+3abcd$.

Two possibilities arise, namely:

(i) If $\Delta \neq 0$, then $w^2 = 0$ is one of the quadrics of the linear ∞^4 system determined by θ and ϕ_i, and therefore all common points of these quadrics must lie in the prime $w = 0$ and hence, on projection, produce no points of the projected Burkhardt locus.

(ii) If $\Delta = 0$, then only four of the quadrics θ and ϕ_i are linearly independent, and they possess a common curve whose nature is sought; under the projection this whole curve is carried into a single point of the projected locus.

Considering -1, a, b, c, d as coordinates in the [4], we see, by (vi), that $\Delta = 0$ is effectively a form of equation of the Burkhardt primal. Our objective now is to obtain a series of lemmas which will lead to the solution of the problem (given in Theorem 2). Such lemmas will not be proved fully, details being left to the reader.

In what follows, write

$$A = a^2+bcd, \quad B = b^2+cda, \quad C = c^2+dab, \quad D = d^2+abc,$$

$$a'^2 = a^2+4bcd, \quad b'^2 = b^2+4cda, \quad c'^2 = c^2+4dab, \quad d'^2 = d^2+4abc.$$

Let N be the net of quadrics

$$\delta\phi_1+\alpha\phi_2+\beta\phi_3+\gamma\phi_4 = 0,$$

where $\alpha A+\beta B+\gamma C+\delta D = 0$.

LEMMA 1. *The quadrics of N are all cones having the same vertex $V \equiv (a, b, c, d, 2)$ [coordinates in the order x, y, z, t, w].*

Verification of this result is easy. It may also be verified that V is the point of intersection of J and the [4] through j. Lemma 1 does not involve $\Delta = 0$.

LEMMA 2. *The section of N by $w = 0$ is a net of quadric surfaces in* [3] *having 6 common points if and only if $\Delta = 0$.*

Taking the section of N by $w = 0$, we obtain the set of four quadrics in [3]

$$\phi_1^* \equiv t^2-ayz-bzx-cxy = 0,$$

$$\phi_2^* \equiv x^2+dyz-cyt+bzt = 0,$$

$$\phi_3^* \equiv y^2+dzx+cxt-azt = 0,$$

$$\phi_4^* \equiv z^2+dxy-bxt+ayt = 0.$$

Let $$W \equiv \delta\phi_1^* + \alpha\phi_2^* + \beta\phi_3^* + \gamma\phi_4^* = 0$$

be the web determined by these quadrics. In addition, let M be the net in W determined by parameters satisfying

$$\alpha A + \beta B + \gamma C + \delta D = 0.$$

It may be verified that the base of M is either the set of 8 *associated* points

$$(a, b, c, \pm d'), \quad (\pm a', b, -c, d), \quad (-a, \pm b', c, d), \quad (a, -b, \pm c', d),$$

or a twisted cubic. (The first hypothesis may easily be shown to be invalid.) In the latter event there must be a cone of the system with its vertex at each base point of M and it is sufficient to find cones for two such points [Wood (**5**)]. On calculation it is found that the condition for each of the 8 associated points to be the vertex of a cone in the system M is $\Delta = 0$. Two of these cones meet, apart from the line joining their vertices, in a twisted cubic curve passing through all 8 points [Todd (**4**)] and therefore lying on all quadrics of the net N.

Various loci, which can only be mentioned here, are perhaps worth a little consideration in terms of the condition $\Delta = 0$. They are

(a) in W the 4 quadric cones like the one for which $\delta = 0$, $\alpha = a$, $\beta = b$, $\gamma = c$, and the quadric given by $\delta = d$, $\alpha = a$, $\beta = b$, $\gamma = c$;

(b) the 4 cubic cones for which $\delta = 0$, t, t, t; $\alpha = x$, 0, $-x$, x; $\beta = y$, y, 0, $-y$; $\gamma = z$, $-z$, z, 0 respectively.

4. Sextic curves of genus 2 on L

Combining Lemmas 1 and 2, we have that the net N in [4] is the system of cones having vertex $V \equiv (a, b, c, d, 2)$ and passing through the cubic conical sheet R joining V to the twisted cubic of Lemma 2.

The locus of points common to the four ϕ_i is the section of R by any quadric, not belonging to N, of the web determined by these four. Such a quadric is ϕ_2. Now ϕ_2 does not pass through V. Hence ϕ_2 cuts R in a sextic curve ${}^2C^6$ of genus 2 [Edge (**2**)], or in a pair of twisted cubics. In this latter case there would be one quadric of the web which is a solid pair. For general values of a, b, c, d, with $\Delta = 0$, no such quadric exists, as can be easily verified. Therefore, we have the lemma:

LEMMA 3. *The four quadrics ϕ_i (with $\Delta = 0$) have in common the ${}^2C^6$ cut on a cubic conical sheet by a quadric of the web determined by them.*

We may now summarize the above remarks.

From the solid j on L the locus L is projected into the Burkhardt primal in the [4] J. In J take a point V. Then the [4] Vj meets L in the locus common to the 5 quadrics θ, ϕ_i determining a linear system H which always contains a net consisting entirely of cones with vertex V. Either H is a system of freedom 4 and contains $w = 0$, in which case Vj meets L in j and nothing else, or, if $\Delta = 0$, H is a web and Vj meets L in j and the ${}^2C^6$. Consequently, to a general point V in J there corresponds nothing (outside the fixed j) on L, but, if V lies on the Burkhardt primal $\Delta = 0$, there corresponds to it the whole ${}^2C^6$ on L.

Summarizing, we therefore have

THEOREM 2. *If $\Delta = 0$, every point on the Burkhardt primal is the projection of some ${}^2C^6$ on L from a solid on L.*

My very deep thanks and appreciation are due to Professor T. G. Room for his generous assistance in the development of the above theory. Finally, I should like to acknowledge the constructive comments offered by the referee in relation to the presentation of this article.

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SOME PROPERTIES OF GRAM MATRICES AND DETERMINANTS

By W. N. EVERITT (*Shrivenham*)

[Received 15 March 1957]

1. In all that follows the symbols E, $E(r)$ will denote measurable linear sets.† Any function which is of integrable square on E is said to belong to *the class of* $L^2(E)$. The symbol $\{\phi\}$ will denote the ordered system of complex-valued functions $\{\phi_1(x),\ldots,\phi_n(x)\}$, where the integer n is fixed throughout the paper.

Let $\{\phi\}$ be defined on a set E and let the following definitions hold:

(*a*) a *linear form* of $\{\phi\}$ is a finite sum

$$\sum_{i=1}^{n} a_i \phi_i(x)$$

with constant complex coefficients a_i one at least of which is not zero;

(*b*) the system $\{\phi\}$ is said to be *linearly dependent over* E if some linear form of $\{\phi\}$ vanishes p.p. on E; in the contrary case $\{\phi\}$ is *linearly independent over* E;

(*c*) if in a linear form of $\{\phi\}$ the coefficient of a particular function, say ϕ_j, is not zero, then the linear form is said to be *dependent on* ϕ_j;

(*d*) the symbol $S(\phi; E)$ will denote the L^2*-span of* $\{\phi\}$ *over* E (or briefly the *span*), and this is defined as the maximum number of linear forms of $\{\phi\}$ which are linearly independent over E *and* which belong to the class of $L^2(E)$. (Roughly speaking $S(\phi; E)$ is the largest number of functions of $L^2(E)$ which can be formed from $\{\phi\}$ by linear combination.)

With these definitions the following results are clear:

(i) $0 \leqslant S(\phi; E) \leqslant n$,

(ii) $S(\phi; E) = n$ if and only if $\{\phi\}$ is linearly independent over E *and* $\phi_i \in L^2(E)$ for $1 \leqslant i \leqslant n$.

2. The problem to be discussed in this note can now be indicated; it appeared originally in the theory of singular differential operators.

In all that follows r, s, and t will denote positive integers.

† By an obvious change in notation we can extend all our results to functions of several variables.

Let the ordered system $\{\phi_1(x),\ldots,\phi_n(x)\}$ be defined over the sequence of sets $\{E(r); r \geqslant 1\}$ and suppose that

(i) $\{\phi\}$ is linearly independent over $E(1)$,

(ii)† $E(r) \subseteq E(r+1) \quad (r \geqslant 1)$,

(iii) $S\{\phi; E(r)\} = n \quad (r \geqslant 1)$.

Conditions (i) and (ii) imply that $\{\phi\}$ is linearly independent over each $E(r)$. The monotonic condition (ii) permits the following definition of the *limit set* E

$$E \equiv \lim_{r\to\infty} E(r) = \bigcup_{r=1}^{\infty} E(r).$$

The problem is to obtain information about $S(\phi; E)$, the span of $\{\phi\}$ over the limit set E, from the behaviour of $\{\phi\}$ over the sequence $\{E(r)\}$. Simple examples serve to show that $S(\phi; E) < n$ is possible in spite of condition (iii) above.

Four of the theorems are stated in the next section and their proofs then given. The last theorem is of a different character, and its statement and proof are left to the last few sections. All of the results obtained depend on the properties of Gram matrices and determinants of the system $\{\phi\}$ over the sequence $\{E(r)\}$. Some of the properties required were given by the author in a previous note (**1**). The general results on matrix theory which are quoted will be found in the book by Mirsky (**2**).

The *Gram matrix*

$$\Gamma\{\phi; E(r)\} \equiv \Gamma\{\phi_1,\ldots,\phi_n; E(r)\}$$

of $\{\phi\}$ on $E(r)$ is defined as the $n\times n$ matrix whose (i,j)-th element is

$$\int_{E(r)} \phi_i \bar{\phi}_j \, dx.$$

The *Gram determinant*

$$\Delta\{\phi; E(r)\} \equiv \Delta\{\phi_1,\ldots,\phi_n; E(r)\}$$

of $\{\phi\}$ on $E(r)$ is defined as $\det \Gamma\{\phi; E(r)\}$.

$\Gamma\{\phi; E(r)\}$ is an hermitian matrix and conditions (i) and (ii) above imply that it is also positive-definite [see (**1**) § 2]. Thus

$$\Delta\{\phi; E(r)\} > 0 \quad (r \geqslant 1).$$

The following definition of μ_t is now possible:

$$\mu_t\{E(r)\} \equiv \frac{\Delta\{\phi_1,\ldots,\phi_t; E(r)\}}{\Delta\{\phi_1,\ldots,\phi_n; E(r)\}} \quad (1 \leqslant t \leqslant n-1).$$

† Strict inclusion for an infinite number of values of r is assumed since otherwise the problem is straightforward.

In (1) the following inequalities were established

$$0 < \Delta\{\phi;\, E(r)\} \leqslant \Delta\{\phi;\, E(r+1)\} \quad (r \geqslant 1),$$

$$0 < \mu_t\{E(r+1)\} \leqslant \mu_t\{E(r)\} \quad (r \geqslant 1;\ 1 \leqslant t \leqslant n-1).$$

3. The monotonic nature of the set functions Δ and μ_t establishes the existence of the limits (finite or infinite) of these functions as $r \to \infty$.

THEOREM 1. *A necessary and sufficient condition that* $\lim \Delta\{\phi;\, E(r)\}$, *as* $r \to \infty$, *be finite is that*

$$S(\phi;\, E) = n.$$

THEOREM 2. *A necessary and sufficient condition that* $\lim \mu_{n-1}\{E(r)\}$, *as* $r \to \infty$, *be positive is that the system* $\{\phi\}$ *should possess a linear form, dependent on* $\phi_n(x)$, *which belongs to the class* $L^2(E)$.

THEOREM 3. *A necessary and sufficient condition that* $\lim \mu_t\{E(r)\}$, *as* $r \to \infty$, *be positive is that the system* $\{\phi\}$ *should possess* $n-t$ *linearly independent linear forms, dependent on* $\{\phi_i;\, t+1 \leqslant i \leqslant n\}$, *which all belong to the class* $L^2(E)$.

Clearly Theorem 2 is a special case of Theorem 3 (when $t = n-1$); however, only the proof of Theorem 2 is given since that for Theorem 3 follows entirely similar, but more lengthy, lines. Theorems 2 and 3 give some idea of how the span $S(\phi;\, E)$ is built up from the individual members of $\{\phi\}$.

For any given order of the functions ϕ_i which make up the system $\{\phi\}$ the behaviour of such terms as $\mu_t\{E(r)\}$ is not, in general, sufficient to determine $S(\phi;\, E)$. For example, if $\phi_n(x)$ is *not* contained in any linear form of $\{\phi\}$ which is of $L^2(E)$, then $\mu_t\{E(r)\} \to 0$ for $1 \leqslant t \leqslant n-1$; this follows from Theorem 2 and the representation

$$\mu_t\{E(r)\} = \prod_{s=t}^{n-1} \frac{\Delta\{\phi_1, \ldots, \phi_s;\, E(r)\}}{\Delta\{\phi_1, \ldots, \phi_{s+1};\, E(r)\}}.$$

The last factor of this product tends to zero for $1 \leqslant t \leqslant n-1$. Thus, in this case all the $\mu_t \to 0$ in spite of the fact that the subset

$$\{\phi_i;\, 1 \leqslant i \leqslant n-1\}$$

may possess linear forms which are $L^2(E)$.

To determine $S(\phi;\, E)$ for the ordered system $\{\phi\}$ define the following set of numbers† $\{\delta_t;\, 1 \leqslant t \leqslant n\}$

$$\delta_t \equiv \begin{cases} \lim\limits_{r\to\infty} [\Delta\{\phi_1;\, E(r)\}]^{-1} & (t = 1), \\ \lim\limits_{r\to\infty} \dfrac{\Delta\{\phi_1, \ldots, \phi_{t-1};\, E(r)\}}{\Delta\{\phi_1, \ldots, \phi_t;\, E(r)\}} & (2 \leqslant t \leqslant n). \end{cases}$$

† These numbers exist by virtue of the inequalities at the end of § 2.

Clearly $$0 \leqslant \delta_t < \infty \quad (1 \leqslant t \leqslant n).$$

This definition leads to the statement of the theorem:

THEOREM 4. *$S(\phi; E)$, the L^2-span of $\{\phi\}$ over the limit set E, is given by the number of the set $\{\delta_t: 1 \leqslant t \leqslant n\}$ which satisfy the condition $\delta_t > 0$.*

4. The proofs of the above theorems require the following lemmas.

In all that follows $\{p_r; r \geqslant 1\}$ and $\{q_r; r \geqslant 1\}$ will denote *strictly* increasing sequences of positive integers.

LEMMA 1. *Let $\{\alpha_i(r); 1 \leqslant i \leqslant n\}$ be n complex-valued sequences, defined for $r \geqslant 1$, where at least one of $|\alpha_i(r)|$ is not bounded as $r \to \infty$. Then there exists a positive integer k $(1 \leqslant k \leqslant n)$ and a sequence $\{p_r; r \geqslant 1\}$ such that*

(i) $\alpha_k(p_r) \neq 0 \quad (r \geqslant 1)$,

(ii) $|\alpha_k(p_r)| \to \infty \quad$ *as* $\quad r \to \infty$,

(iii) *if we define*

$$\beta_i(r) \equiv \alpha_i(p_r)[\alpha_k(p_r)]^{-1} \quad (1 \leqslant i \leqslant n),$$

then there exists an absolute constant M such that

$$|\beta_i(r)| \leqslant M \quad \text{for} \quad r \geqslant 1 \quad \text{and} \quad 1 \leqslant i \leqslant n.$$

Proof. This is by induction on n. The result is clearly true when $n = 1$.

Let the lemma hold for any set with $n-1$ members. If every member of the set $\{\alpha_i(r); 1 \leqslant i \leqslant n-1\}$ is bounded, then, by hypothesis, $\alpha_n(r)$ is not bounded and a sequence $\{p_r\}$ exists for which (i) and (ii) above hold with $k = n$; condition (iii) then follows.

Alternatively, if one of $\alpha_i(r)$ $(1 \leqslant i \leqslant n-1)$ is not bounded, let k' and $\{p'_r\}$ be the integer and sequence for which the lemma is true for this set of $n-1$ members. Define

$$\gamma(r) \equiv \alpha_n(p'_r)[\alpha_{k'}(p'_r)]^{-1} \quad (r \geqslant 1);$$

if $\liminf|\gamma(r)|$, as $r \to \infty$, is finite, then there is a *sub-sequence* of $\{p'_r\}$, say $\{p_r\}$, such that $\gamma(p_r)$ tends to a finite limit; we then put $k = k'$ and choose this sequence $\{p_r\}$ for the lemma to hold. On the other hand, if $\liminf|\gamma(r)|$ is infinite, then there is a *sub-sequence* of $\{p'_r\}$, say $\{p_r\}$, such that (i) and (ii) above hold for $\alpha_n(p_r)$ and $\lim|\gamma(p_r)| = \infty$; we now choose $k = n$ and this sequence $\{p_r\}$ for the lemma to hold in this case.

LEMMA 2. *Let the system $\{\phi_1(x), \ldots, \phi_n(x)\}$ and the sequence $\{E(r)\}$ satisfy*

the conditions (i), (ii), *and* (iii) *of* § 2. *Let* $\{\alpha_i(r); 1 \leqslant i \leqslant n\}$ *be* n *complex-valued sequences. Define*

$$\psi(r; x) \equiv \sum_{i=1}^{n} \alpha_i(r)\phi_i(x) \quad (x \in E(r);\ r \geqslant 1)$$

and suppose that the following two hypotheses hold:

(α) *there exists a constant* M *such that*

$$\int_{E(r)} |\psi(r; x)|^2\, dx \leqslant M \quad \text{for} \quad r \geqslant 1,$$

(β) *for at least one value of* i, *say* $i = j$,

$$\liminf_{r\to\infty} |\alpha_j(r)| > 0,$$

then

(a) *there exists a constant* K *such that*

$$|\alpha_i(r)| \leqslant K \quad \text{for } 1 \leqslant i \leqslant n \quad \text{and} \quad r \geqslant 1,$$

(b) *there exists a sequence* $\{p_r; r \geqslant 1\}$ *and a set of complex numbers* $\{a_i; 1 \leqslant i \leqslant n\}$ *such that*

$$\alpha_i(p_r) \to a_i \quad (r \to \infty;\ 1 \leqslant i \leqslant n)$$

with

$$|a_i| \leqslant K, \qquad a_j \neq 0$$

and

$$\sum_{i=1}^{n} a_i \phi_i(x) \in L^2(E), \tag{4.1}$$

(c)

$$\liminf_{r\to\infty} \int_{E(r)} |\psi(r; x)|^2\, dx > 0.$$

Proof. This is in two parts. In the first result (a) is assumed to hold, and (b) and (c) then follow. Using this information we then show that (a) is a consequence of the hypotheses (α) and (β).

(A) Assume then that (a) is part of the hypothesis. The first part of (b) then follows by a repeated application of the Bolzano–Weierstrass theorem† on limit points; $a_j \neq 0$ by virtue of hypothesis (β). To establish (4.1) we use (α) above; for every positive integer t, using dominated convergence, we have

$$\begin{aligned}\int_{E(t)} \Big|\sum_{i=1}^{n} a_i \phi_i(x)\Big|^2\, dx &= \lim_{r\to\infty} \int_{E(t)} |\psi(p_r; x)|^2\, dx \\ &\leqslant \limsup_{r\to\infty} \int_{E(p_r)} |\psi(p_r; x)|^2\, dx \leqslant M.\end{aligned}$$

Now let $t \to \infty$, and (4.1) follows.

† See, for example, the proof of the 'selection' theorem in (4) 25.

Finally, if (c) is false, we have a sequence $\{q_r; r \geqslant 1\}$ such that

$$\lim_{r\to\infty} \int_{E(q_r)} |\psi(q_r; x)|^2 \, dx = 0,$$

and, if we apply to this result the above arguments for proving result (b), it follows that

$$\int_{E(1)} \left| \sum_{i=1}^{n} a_i' \phi_i \right|^2 dx = 0$$

for some set $\{a_i'; 1 \leqslant i \leqslant n\}$ with $a_j' \neq 0$. This however contradicts (i) of § 2; thus (c) must hold.

(B) In this second part we suppose that only (α) and (β) hold and assume that (a) above is false; a contradiction then follows. Thus at least one member of the set $\{\alpha_i(r); 1 \leqslant i \leqslant n\}$ is unbounded as $r \to \infty$, and consequently let k, $\{p_r\}$, $\beta_i(r)$ have the same meaning as in Lemma 1. Then for $r \geqslant 1$ we have $\beta_k(r) = 1$ and, from (α) above,

$$\int_{E(p_r)} \left| \sum_{i=1}^{n} \beta_i(r)\phi_i \right|^2 dx \leqslant M|\alpha_k(p_r)|^{-2}. \tag{4.2}$$

Thus all the conditions for the arguments used in (A) above are satisfied for the set $\{\beta_i(r)\}$ (with $j = k$) and the sequence $\{E(p_r)\}$, so that the result (c) in particular must hold for the function $\sum \beta_i(r)\phi_i$. On the other hand, from (4.2) and (ii) of Lemma 1,

$$\limsup_{r\to\infty} \int_{E(p_r)} \left| \sum_{i=1}^{n} \beta_i(r)\phi_i \right|^2 dx = 0,$$

and this gives a contradiction. Thus (a) follows from (α) and (β), and the proof of the lemma is complete.

The following corollary is also needed:

COROLLARY. *Let $\psi(r; x)$ be defined as for Lemma 2 but suppose that only the hypothesis (β) holds, then*

$$\liminf_{r\to\infty} \int_{E(r)} |\psi(r; x)|^2 \, dx > 0.$$

If the above conclusion is false, then arguing as in the proof of the above lemma we obtain a contradiction.

5. The proof of Theorem 1 of § 3 can now be given.

If $S(\phi; E) = n$, then, from (ii) of § 1 we have $\phi_i \in L^2(E)$ $(1 \leqslant i \leqslant n)$. Clearly then, by the Cauchy–Schwarz inequality, each element of $\Delta\{\phi; E(r)\}$ tends to a finite limit as $r \to \infty$, and Δ itself will tend to a finite positive limit. Suppose now that the limit of $\Delta\{\phi; E(r)\}$ is finite

and yet $S(\phi; E) < n$; then, for at least one value of i, we have $\phi_i \notin L^2(E)$. Clearly we can suppose that $i = 1$.

Let the symbol ϕ denote the column matrix† $(\phi_1(x),..., \phi_n(x))^T$.

For each $r \geqslant 1$ we apply the Schmidt orthogonalization process to the system $\{\phi\}$ over $E(r)$ [for details of this process see Courant and Hilbert (3) 50]. Thus we have a sequence $\{T(r); r \geqslant 1\}$ of lower triangular matrices with unity along the principal diagonal, i.e.

$$T(r) = [t_{ij}(r)] \text{ (say)} \qquad (1 \leqslant i, j \leqslant n)$$

$$t_{ii}(r) = 1, \qquad t_{ij}(r) = 0 \quad (j > i; r \geqslant 1),$$

such that, if the system of functions $\{\psi_i(r; x);\ 1 \leqslant i \leqslant n\}$ is defined by the matrix equation $\psi(r) = T(r)\phi$, where

$$\psi(r) = (\psi_1(r; x),..., \psi_n(r; x))^T,$$

then this system $\{\psi_i(r; x)\}$ is orthogonal over $E(r)$.

We now apply the transformation $\psi(r) = T(r)\phi$ to the Gram determinant $\Delta\{\phi; E(r)\}$ to obtain, for $r \geqslant 1$,

$$\Delta\{\phi; E(r)\} = |\det T(r)|^{-2}\Delta\{\psi(r); E(r)\} \quad \text{[see (1) Lemma 1]}$$

$$= \prod_{i=1}^{n} \int_{E(r)} |\psi_i(r; x)|^2\, dx, \quad \text{since } \det T(r) = 1, \tag{5.1}$$

$$= \int_{E(r)} |\phi_1(x)|^2\, dx \prod_{i=2}^{n} \int_{E(r)} |\psi_i(r; x)|^2\, dx \tag{5.2}$$

since $\qquad \psi_1(r; x) = \phi_1(x) \quad \text{for} \quad x \in E(r).$

From the corollary to Lemma 2 we have

$$\liminf_{r\to\infty} \int_{E(r)} |\psi_i(r; x)|^2\, dx > 0 \quad (2 \leqslant i \leqslant n)$$

since $t_{ii}(r) = 1$ $(1 \leqslant i \leqslant n)$ for $r \geqslant 1$ is equivalent to hypothesis (β) for each $\psi_i(r; x)$.

Clearly then, since $\phi_1(x) \notin L^2(E)$, (5.2) now implies

$$\lim_{r\to\infty} \Delta\{\phi; E(r)\} = \infty.$$

This is a contradiction and we must have $S(\phi; E) = n$.

6. In this section the proof of Theorem 2 is given.

Suppose, to begin with, that the limit of $\mu_{n-1}\{E(r)\}$ is positive. We

† T indicates a transposed matrix.

again apply the Schmidt process to the system $\{\phi\}$ over $E(r)$ and write $\psi(r) = T(r)\phi$ as in § 5. The triangular nature of $T(r)$ also gives

$$(\psi_1(r;x),\ldots,\psi_{n-1}(r;x))^T = S(r)(\phi_1(x),\ldots,\phi_{n-1}(x))^T, \tag{6.1}$$

where $$S(r) = [t_{ij}(r)] \quad (1 \leqslant i, j \leqslant n-1).$$

Clearly $$\det T(r) = \det S(r) = 1.$$

We now use the result (5.1) and in a similar manner apply the transformation (6.1) to $\Delta\{\phi_1,\ldots,\phi_{n-1};\, E(r)\}$ to obtain

$$\mu_{n-1}\{E(r)\} = \left[\int_{E(r)} |\psi_n(r;x)^2\, dx\right]^{-1}.$$

Thus, by hypothesis,

$$\lim_{r\to\infty} \int_{E(r)} |\psi_n(r;x)|^2\, dx = \left[\lim_{r\to\infty} \mu_{n-1}\{E(r)\}\right]^{-1} < \infty,$$

and, since $$\psi_n(r;x) = \sum_{i=1}^{n} t_{ni}(r)\phi_i(x) \quad \text{with} \quad t_{nn}(r) = 1,$$

all the conditions and hypotheses (α) and (β) (with $j = n$) of Lemma 2 are satisfied by $\psi_n(r;x)$. From (c) of that lemma we deduce the existence of a linear form, say $\chi(x)$, of $\{\phi\}$, where

$$\chi(x) = \sum_{i=1}^{n} a_i \phi_i(x) \quad (a_n = 1) \tag{6.2}$$

such that $\chi(x) \in L^2(E)$.

To prove Theorem 2 in the opposite direction suppose that a linear form $\chi(x)$, given by (6.2), exists such that $\chi \in L^2(E)$. We require the matrix $S(r)$ above and the system $\{\psi_i(r;x);\ 1 \leqslant i \leqslant n-1\}$. We can also write

$$(\psi_1(r;x),\ldots,\psi_{n-1}(r;x),\chi(x))^T = P(r)(\phi_1,\ldots,\phi_n)^T \quad (r \geqslant 1), \tag{6.3}$$

where, with partitioned matrices, the $n\times n$ matrix $P(r)$ is given by

$$P(r) = \begin{bmatrix} S(r) & 0 \\ a & 1 \end{bmatrix}, \quad a = (a_1,\ldots,a_{n-1}).$$

Clearly $$\det S(r) = \det P(r) = 1 \quad (r \geqslant 1).$$

The transformation (6.1) is applied to the numerator of $\mu_{n-1}\{E(r)\}$, as before. The transformation (6.3) is applied to the denominator and from the resulting determinant, namely

$$\Delta\{\psi_1,\ldots,\psi_{n-1},\chi;\, E(r)\},$$

the factor $$\prod_{i=1}^{n-1} \int_{E(r)} |\psi_i(r;x)|^2\, dx$$

is removed to cancel with that in the numerator. It can now be seen that

$$\mu_{n-1}\{E(r)\} = \left\{\det\begin{bmatrix} I_{n-1} & \bar{\zeta}(r)^T \\ \zeta(r) & \eta(r) \end{bmatrix}\right\}^{-1}, \tag{6.4}$$

where I_{n-1} is the $(n-1)\times(n-1)$ unit matrix and

$$\zeta(r) = \text{row}\left\{\int_{E(r)} \chi(x)\bar{\psi}_i(r;x)\,dx\left[\int_{E(r)} |\psi_i(r;x)|^2\,dx\right]^{-\frac{1}{2}}\right\} \quad (1 \leqslant i \leqslant n-1),$$

$$\eta(r) = \int_{E(r)} |\chi(x)|^2\,dx.$$

We know from § 2 that $\lim \mu_{n-1}\{E(r)\}$ is finite and non-negative; to prove that this limit is positive we need only prove that each element of the determinant in (6.4) is bounded as $r \to \infty$. This, however, follows at once on inspection of $\zeta(r)$ and $\eta(r)$, use of the assumption that $\chi \in L^2(E)$, and the Cauchy–Schwarz inequality. This completes the proof.

As stated in § 3 the proof of Theorem 3 is omitted since this is similar to the proof of Theorem 2.

7. The proof of Theorem 4 is by induction on n, the number of functions in the system $\{\phi\}$.

Clearly the result is true when $n = 1$ by definition of δ_1. Now suppose that the result is true for all systems with $n-1$ members where $n > 1$. Then the span of the system $\{\phi_i;\ 1 \leqslant i \leqslant n-1\}$ is given by the number of δ_t $(1 \leqslant t \leqslant n-1)$ which are positive.

Consider δ_n; if this is positive, then by Theorem 2 there is a linear form, of $L^2(E)$, of the system $\{\phi\}$ which is dependent on ϕ_n and this is linearly independent of any form, of $L^2(E)$, which depends solely on $\{\phi_i(x);\ 1 \leqslant i \leqslant n-1\}$.

Conversely, if δ_n is zero, then no independent form, of $L^2(E)$, can be added to those already existing (if any). Thus the result is true in general.

8. The last theorem depends on a property of the characteristic roots of hermitian matrices. Let the characteristic roots of the Gram matrix $\Gamma\{\phi;\ E(r)\}$ be designated by $\{\lambda_i(r);\ 1 \leqslant i \leqslant n\}$. Clearly, since Γ is positive-definite, $\lambda_i(r) > 0$ for all i and r.

Suppose, without loss of generality, that, for all $r \geqslant 1$,

$$\lambda_i(r) \leqslant \lambda_{i+1}(r) \quad (1 \leqslant i \leqslant n-1). \tag{8.1}$$

Then this notation and the monotonic condition (ii) of § 2, i.e.

$$E(r) \subseteq E(r+1),$$

imply the following monotonic property of the characteristic roots valid for $r \geqslant 1$,

$$\lambda_i(r) \leqslant \lambda_i(r+1) \quad (1 \leqslant i \leqslant n). \tag{8.2}$$

This last result follows from a theorem given by Courant and Hilbert [see (3) 33].

From (8.1) and (8.2) the following definition of the non-negative integer s as the number of $\lambda_i(r)$ which tend to finite limits as $r \to \infty$ is justified:

$$\lim_{r\to\infty} \lambda_i(r) \begin{cases} < \infty & (i \leqslant s), \\ = \infty & (i > s). \end{cases} \tag{8.3}$$

Clearly $0 \leqslant s \leqslant n$.

THEOREM 5. *With the definition of $S(\phi; E)$, the L^2-span of $\{\phi\}$ over E, given in* § 1 *and the above definition of s it follows that*

$$S(\phi; E) = s.$$

9. The next three sections are taken up with the proof of Theorem 5.

Since $\Gamma\{\phi; E(r)\}$ is hermitian, there is, for $r \geqslant 1$, an $n \times n$ *unitary* matrix $U(r)$ such that†

$$U(r)\Gamma\{\phi; E(r)\}U(r)^* = \operatorname{diag}(\lambda_1(r), \ldots, \lambda_n(r)). \tag{9.1}$$

Now define, for $r \geqslant 1$, the system $\{\psi_i(r; x);\ 1 \leqslant i \leqslant n\}$ by the matrix equation $\psi(r) = U(r)\phi$, where the matrix notation of § 5 is used again. If we now transform the Gram matrix $\Gamma\{\phi; E(r)\}$ by this transformation, we obtain [see (1) Lemma 1]

$$\Gamma\{\psi(r); E(r)\} = U(r)\Gamma\{\phi; E(r)\}U(r)^*. \tag{9.2}$$

Comparison of (9.1) and (9.2) now gives, for $r \geqslant 1$,

$$\int_{E(r)} \psi_i(r; x)\bar{\psi}_j(r; x)\, dx = \delta_{ij}\lambda_i(r) \quad (1 \leqslant i, j \leqslant n). \tag{9.2a}$$

Let the elements of $U(r)$ be $\alpha_{ij}(r)$; since $U(r)$ is unitary, the following results hold,‡ for $r \geqslant 1$,

$$|\alpha_{ij}(r)| \leqslant 1 \quad (1 \leqslant i, j \leqslant n), \tag{9.3}$$

$$\det U(r) = 1. \tag{9.4}$$

From (9.3) and repeated use of the Bolzano–Weierstrass principle there is a sequence $\{p_r;\ r \geqslant 1\}$ and a set of complex numbers $\{a_{ij}\}$ such that

$$\lim_{r\to\infty} \alpha_{ij}(p_r) = a_{ij}, \qquad |a_{ij}| \leqslant 1 \quad (1 \leqslant i, j \leqslant n),$$

and, from (9.4),

$$\det[a_{ij}] = 1. \tag{9.5}$$

† An asterisk will denote the conjugate transpose of a matrix.

‡ The positive sign can be taken on the right-hand side of (9.4) without loss of generality.

Now define the system of functions $\{\chi\}$ by

$$\chi_i(x) \equiv \sum_{j=1}^{n} a_{ij}\phi_j(x) \quad (1 \leqslant i \leqslant n;\ x \in E). \tag{9.6}$$

From (9.5) the system $\{\chi\}$ is linearly independent over $E(1)$ and hence over E; also, for $1 \leqslant i \leqslant n$ and $x \in E$,

$$\psi_i(p_r; x) = \sum_{j=1}^{n} \alpha_{ij}(p_r)\phi_j(x) \to \chi_i(x) \quad \text{as } r \to \infty. \tag{9.7}$$

10. In this section it is shown that

$$S(\phi; E) \geqslant s. \tag{10.1}$$

Let $s = 0$; then (10.1) follows at once in this case. Suppose then that $s > 0$; from (8.3), (9.2 a) and for the sequence $\{p_r\}$ of § 9,

$$\lim_{r\to\infty} \int_{E(p_r)} |\psi_i(p_r; x)|^2\, dx = \lim_{r\to\infty} \lambda_i(p_r) < \infty \quad (1 \leqslant i \leqslant s). \tag{10.2}$$

Also for each value of i in $1 \leqslant i \leqslant n$ there exists at least one value of j such that

$$\liminf_{r\to\infty} |\alpha_{ij}(p_r)| = \lim_{r\to\infty} |\alpha_{ij}(p_r)| = |a_{ij}| > 0 \tag{10.3}$$

must hold; otherwise (9.5) is violated. Thus (10.2) and (10.3) imply that the hypotheses (α) and (β) of Lemma 2 are satisfied for each of the functions $\psi_i(p_r; x)$ $(1 \leqslant i \leqslant s)$, and from this and (9.7) it follows that $\chi_i(x) \in L^2(E)$ $(1 \leqslant i \leqslant s)$. Hence, there are at least s linearly independent linear forms of $\{\phi\}$ which belong to the class $L^2(E)$, and (10.1) now holds for all possible values of s.

11. In this section the converse of (10.1) is shown to hold, i.e.

$$S(\phi; E) \leqslant s. \tag{11.1}$$

We can suppose that

$$S \equiv S(\phi; E) > 0, \qquad s < n \tag{11.1 a}$$

since, if either of these does not hold, (11.1) follows at once. Thus there are S linearly independent linear forms, say $\{\theta_i(x);\ 1 \leqslant i \leqslant S\}$, of $\{\phi\}$ which belong to $L^2(E)$. Since, from § 9,

$$\phi = U(r)^{-1}\psi(r),$$

there is a set $\{\beta_{ij}(p_r)\}$, defined on the sequence $\{p_r\}$ of § 9, such that, for $1 \leqslant i \leqslant S$,

$$\theta_i(x) = \sum_{j=1}^{n} \beta_{ij}(p_r)\psi_j(p_r; x) \quad (r \geqslant 1). \tag{11.2}$$

The orthogonal property (9.2 a) of $\{\psi_i(p_r; x)\}$ over $E(p_r)$ leads to the determination of the β_{ij},

$$\beta_{ij}(p_r) = \int_{E(p_r)} \theta_i(x)\bar{\psi}_j(p_r; x)\, dx \Big[\int_{E(p_r)} |\psi_j(p_r; x)|^2\, dx\Big]^{-1},$$

so that, for $1 \leqslant i \leqslant S$, $1 \leqslant j \leqslant n$, and $r \geqslant 1$,

$$|\beta_{ij}(p_r)|^2 \leqslant \int_{E(p_r)} |\theta_i|^2 \, dx \left[\int_{E(p_r)} |\psi_j(p_r; x)|^2 \, dx \right]^{-1}. \tag{11.3}$$

Since $\theta_i(x) \in L^2(E)$, it follows from (11.3) that there is a constant L such that, for the above values of i and j,

$$|\beta_{ij}(p_r)| \leqslant L \quad (r \geqslant 1).$$

Again from the Bolzano–Weierstrass principle there is a sub-sequence of $\{p_r\}$, say $\{q_r; r \geqslant 1\}$, and a set $\{b_{ij}\}$ such that

$$\lim_{r \to \infty} \beta_{ij}(q_r) = b_{ij} \quad (|b_{ij}| \leqslant L).$$

From (9.7) and the use of (11.2) for the sequence $\{q_r\}$ we have, on letting $r \to \infty$,

$$\theta_i(x) = \sum_{j=1}^{n} b_{ij} \chi_j(x) \quad (1 \leqslant i \leqslant S).$$

Since, by hypothesis, the system $\{\theta_i; 1 \leqslant i \leqslant S\}$ is linearly independent, the rank of the $S \times n$ matrix $[b_{ij}]$ *must* be S.

By (11.1 a) the set of j for which $s+1 \leqslant j \leqslant n$ is not empty; from the definition of s and (9.2 a), the integrals

$$\int_{E(p_r)} |\psi_j(p_r; x)|^2 \, dx \quad (s+1 \leqslant j \leqslant n)$$

all tend to infinity with r. This implies, from the inequalities (11.3), that

$$b_{ij} = 0 \quad (1 \leqslant i \leqslant S; \, s+1 \leqslant j \leqslant n).$$

Thus $n-s$ columns of $[b_{ij}]$ are all zero; but, since the rank of $[b_{ij}]$ is S, there can be *at most* $n-S$ zero columns, i.e.

$$n-s \leqslant n-S, \qquad S \leqslant s.$$

Thus (11.1) holds in all cases.

From (10.1) and (11.1) the proof of Theorem 5 is complete.

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ONE-SIDED INEQUALITIES FOR INTEGRAL QUADRATIC FORMS

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In recent years several writers have investigated the lower bound of the positive, or of the non-negative values of f, $-f$, or $|f|$, f being an indefinite quadratic form, with *real* coefficients, in integral variables $x_1,\ldots, x_k$ (not all zero). Estimates for $k \leqslant 4$ are given in (1), (4), (5), (7). For $k \geqslant 5$ it seems likely that all the lower bounds in question are zero if the ratios of the coefficients are not all rational; and Oppenheim (6) has shown that in this case the lower bounds all vanish if any one of them does so. Davenport (2) has shown that the lower bound of the non-negative values of $|f|$ is zero if† $|s| \leqslant k-37$, where s is the signature of f.

In view of the last two results, it seems of interest to consider the case in which the ratios of the coefficients are all rational. In this case nothing is lost by supposing that the coefficients are integral, and that f is primitive. I make these assumptions throughout this paper, and also assume that f is non-singular: that is, that its discriminant defined below does not vanish. I consider the lower bound of the strictly positive values of f; this lower bound is clearly an attained minimum, and I denote it by $\min^+ f$.

The problem was suggested to me by Dr. Chalk. For small values of the signature the method I use is not new and the theorem I prove would follow from some unpublished work of his which Professor Oppenheim has kindly communicated to me. Miss D. M. E. Foster has shown me her proof, for $|s| \leqslant 8$, f indefinite, of my estimate (7) below, with the constant 1, $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$ replaced by 2. She has used her result in some work on inhomogeneous forms which she intends to publish shortly.

To avoid restricting the signature I use a simple deduction (Lemma 3 below) from the classical theory of the genus, which is of some interest in itself.

1. I use non-Gaussian notation, in which the matrix $A = A(f)$ of the

† Professor Davenport tells me that, in a paper to appear in *Proc. London Math. Soc.*, he has weakened this condition to $|s| \leqslant k-16$.

form f is $(\partial^2 f/\partial x_i \partial x_j)$ $(i, j = 1,..., k)$ and the discriminant $d = d(f)$ is defined by

$$d(f) = (-1)^{\frac{1}{2}k} \det A \ (k \text{ even}); \ \tfrac{1}{2}(-1)^{\frac{1}{2}(k-1)} \det A \ (k \text{ odd}). \tag{1}$$

Since A is congruent modulo 2 to a skew matrix, which for odd k is singular, it is clear that d is always integral. The Gaussian determinant

$$\Delta(f) = \det(\tfrac{1}{2}A) = (-\tfrac{1}{4})^{[\frac{1}{2}k]} d(f) \tag{2}$$

is not always integral. The signature $s = s(f)$ of f is the (unique) integer, easily seen to satisfy

$$|s| \leqslant k, \qquad s \equiv k \pmod 2, \qquad (-1)^{[\frac{1}{2}s]} = \operatorname{sgn} d, \tag{3}$$

such that f can be expressed as $\pm\xi_1^2 \pm ... \pm \xi_k^2$, the ξ_i being real linear forms in the x_i, and the number of positive signs being $\frac{1}{2}(k+s)$.

The constant in the inequality (7) below is that needed for a form with $\min^+ f = 1$ and

$$|d| = 1, 2, 3, 4 \text{ according as } \pm s \equiv 0 \text{ or } 1, 3, 2, 4 \pmod 8. \tag{4}$$

For every pair k, s satisfying $(3)_1$ and $(3)_2$ there exists a form

$$F(k, s) = F(k, s; x_1,..., x_k)$$

satisfying (4). $F(k, s)$ may be defined as follows. For $1 \leqslant s = k \leqslant 8$, we take $F(k, s) = F(k, k)$ to be an extreme positive form. We write

$$F(1,1) = x_1^2,$$

$$F(2,2) = x_1^2 + x_1 x_2 + x_2^2,$$

$$F(3,3) = x_1^2 + x_2^2 + x_3^2 + x_2 x_3 + x_3 x_1 + x_1 x_2,$$

$$F(4,4) = F(5,5; x_1,..., x_4, 0),$$

$$F(5,5) = (x_1 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5)^2 + (x_2 + \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5)^2 + (x_3 + \tfrac{1}{2}x_4)^2 + \tfrac{1}{4}x_4^2 + \tfrac{1}{2}x_5^2,$$

$$F(6,6) = F(8,8; x_1,..., x_5, x_6, x_6),$$

$$F(7,7) = F(8,8; x_1,..., x_7, 0),$$

$$F(8,8) = \tfrac{1}{4} \sum_{i=1}^{4} \Big\{ x_i^2 + \Big(2x_{i+4} + \sum_{\substack{j=1 \\ j \neq i}}^{4} x_j \Big)^2 \Big\}.$$

We take $F(0, 0)$, for convenience, to be identically zero and complete the definition of $F(k, k)$ inductively by writing, for $k \geqslant 9$,

$$F(k, k) = F(8, 8; x_1,..., x_8) + F(k-8, k-8; x_9,..., x_k).$$

Now, for $0 \leqslant s < k$, we write

$$F(k, s) = x_1 x_2 + ... + x_{k-s-1} x_{k-s} + F(s, s; x_{k-s+1},..., x_k)$$

and, for $s < 0$, $\qquad F(k, s) = -F(k, |s|)$.

We shall see that the three ternary forms given by

$$f = hx_1x_2 - x_3^2, \qquad h = 4, 8, 16, \qquad \min^+ f = 3, 4, 7, \tag{5}$$

with $d(f) = -h^2$, and the quaternary form

$$f = 3x_1x_2 - x_3^2 - 3x_3x_4 - 3x_4^2, \quad d(f) = -27, \quad \min^+ f = 2, \tag{6}$$

are exceptional. The values stated for $\min^+ f$, for these exceptional f, are easily found by considering the possibilities for f modulo h or 3.

2. I now state the result, and note some cases for which it is already known.

THEOREM. *Suppose that f is an integral, primitive, non-singular quadratic form, which if definite is positive and of rank $k \leqslant 8$. Suppose also, in case $k = 3$ or 4, that f is not equivalent to any of the forms $(5)_1$, $(5)_3$, (6). Then*

$$\frac{(\min^+ f)^k}{|d(f)|} \leqslant \begin{Bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{Bmatrix} \quad \text{if } s(f) \equiv \begin{Bmatrix} 0 \quad \text{or } \pm 1 \\ \pm 3 \\ \pm 2 \\ 4 \end{Bmatrix} \pmod 8. \tag{7}$$

The sign of equality is necessary for every pair k, s, but only if f satisfies (4) or is equivalent to the form $(5)_2$.

For positive forms with $k \leqslant 8$ the theorem is merely a restatement of well-known classical results, which are included to simplify the inductive argument. This may be seen by substituting from (2) in (7), which then takes a more familiar but less elegant shape.

The need for the sign of equality may be seen by considering the forms $F(k, s)$ defined in the last section.

For $k = 2$, $s = 0$, the theorem is also well known [see, e.g. **(4)** 329, Lemmas 1, 2, 3]. For $k = 3$, $s = 1$, the result has been proved by Davenport [see **(4)** 328 for a stronger result]. For $k = 3$, $s = -1$ [see **(1)** 186, Theorem 1] if f is not a zero form. If $k = 4$, then the left-hand member of (7) is $J_1/16$ in the notation of **(5)**, which gives the result for $s = 0, 2$ [418, Theorems 1, 2]. For $k = 4$, $s = -2$, the result is known if f is not a zero form [see **(1)** 186, Theorem 2].

By a classical theorem of Meyer's, every indefinite form with $k \geqslant 5$ is a zero form. So, if convenient, we may assume that f is a zero form; also, in case $k \leqslant 4$, that k, $s = 3, -1$ or $4, -2$. And all that remains to be proved is that (7) holds, and that there is equality in (7) only if (4) holds: that is, only if $\min^+ f = 1$.

3. We begin with the argument for forms with numerically large signature. We first prove a lemma:

LEMMA 1. *If f is indefinite and $k \geqslant 4$, then f represents every non-zero integer n not excluded by congruence considerations: that is, every $n \neq 0$ such that*

$$f(x_1,\ldots,x_k) \equiv n \pmod m \tag{8}$$

is soluble for every $m \neq 0$. If $k = 3$ and $s = -1$, then this result still holds for positive n.

Proof. For $k \geqslant 4$ see (9). For $k = 3$, there may be integers n for which (8) is soluble for every $m \neq 0$, but $f = n$ is insoluble. Every such integer, however (for indefinite f), satisfies certain conditions [(3) 594, Theorem 2], one of which is $nd > 0$. Now with $k = 3$, $s = -1$, we have $d < 0$ by (3), whence the result.

We shall say that two forms f, f' are *equivalent modulo m* if there is an integral unimodular transformation, with matrix T, such that the transform of f by T satisfies

$$f^T \equiv f' \pmod m, \tag{9}$$

identically in the variables. If this is true, with T depending on m, for every $m \neq 0$, we shall say that f, f' are *equivalent to every modulus*. If so, then it is clear that f' may replace f without affecting the solubility of (8); and also that f, f' have the same rank and discriminant. We shall show that it suffices to consider a finite set of values of m in (9). Clearly, it suffices to take m to be a prime p^r, and we show that primes p not dividing $2d$, and values of r large in relation to d, need not be considered.

LEMMA 2. *Suppose that f, f' have the same rank k and discriminant d, and are equivalent modulo p for each p dividing $2d$ and suitable $r = r(p, d)$. Then f, f' are equivalent to every modulus.*

Proof. This is well known [see, e.g. (8) 191–3, Lemma 5.1, Theorems 5.2, 5.3. In (8) rational transformations with denominators prime to p are used, but this is clearly immaterial.] The results quoted show that f is equivalent modulo p^r to a canonical form which is determined uniquely by k, d if $p \nmid 2d$. If $p \mid 2d$ and r is large enough, the canonical form has only to be trivially modified, by altering its last term in a way easily determined by comparison of discriminants, so as to make it equivalent to f modulo p^{r+1}.

With the notation of the present paper, this lemma could be put into the sharper and more elegant shape: *If $d(f) = d(f')$, $k(f) = k(f')$, and f, f' are equivalent modulo $d(f)$, then they are equivalent to every modulus.*

Now we deduce our main lemma:

LEMMA 3. *If* $|s'| \leqslant k(f)$ *and* $s' \equiv s(f) \pmod 8$, *then there exists a form* f' *with*

$$k(f') = k(f), \qquad d(f') = d(f), \qquad s(f') = s'$$

which is equivalent to f *to every modulus.*

Proof. Pall [(8) 194, Theorem 5.5] has stated the necessary and sufficient conditions for the existence of f with prescribed k, d, s and prescribed congruence properties: that is, with f equivalent modulo p^r to some given form, say f_p, for each p dividing $2d$, with $r = r(p,d)$ as in Lemma 2. By Lemma 2, it suffices to show that these conditions remain satisfied when k, d and the congruence properties are left unaltered, but $s(f)$ is replaced by s'. Now the conditions in question do not involve s explicitly, nor implicitly except in so far as they depend on a certain rational invariant $c_\infty(f)$. This invariant $c_\infty = c_\infty(f)$ is, in our notation, $(-1)^{k(k-s-2)}$ [see (8) 191, $(3)_3$]. Thus c_∞ depends only on the residues of k and s modulo 8, which completes the proof.

We apply Lemma 3 with $0 \leqslant s' \leqslant 7$, whenever this choice is consistent with $|s'| \leqslant k$ and does not imply $s' = s(f)$. Then f' is either indefinite or positive, and $k \leqslant 7$ in the latter case. From Lemma 3 it is clear that (8) is soluble for every m if f' represents n, in particular if $n = \min^+ f'$; then Lemma 1 shows that $\min^+ f \leqslant \min^+ f'$. Now, with $s' \equiv s(f) \pmod 8$, the constant on the right of (7) is the same for f as for f'; thus the theorem is true for f if it is true for f'.

A little consideration now shows that, if the theorem is proved for $0 \leqslant s \leqslant 7$, it will follow in all other cases, except those in which the foregoing choice of s' is impossible, making $s' > k$. These are the cases $k = 3$, $s = -1$; $k = 4$, $s = -2$; $k = 5$, $s = -1$.

4. The rest of the proof is based on a lemma:

LEMMA 4. *Every zero form with* $k \geqslant 3$ *is equivalent to one of the type*

$$f = x_2(a_{12}x_1 + a_{22}x_2 + \ldots + a_{k2}x_k) + g(x_3, \ldots, x_k), \tag{10}$$

where g *is a (possibly imprimitive)* $(k-2)$*-ary form, and* $a_{12} > 0$.

Proof. Suppose that we are given a zero form, which without loss of generality we may take to have leading coefficient 0. Then, calling the variables y_i for convenience, we may write it as $y_1 L + \psi$, where L is a linear, and ψ a quadratic form in $y_2, \ldots, y_k$. Then L cannot vanish, or the form would be degenerate; so let $a_{12} > 0$ be the greatest common divisor of the coefficients of L. Then we may write $L = a_{12} l$, where the coefficients of l have g.c.d. 1. We now transform $y_2, \ldots, y_k$ into new variables $x_2, \ldots, x_k$ in such a way that l goes into x_2. This is possible by

a well-known theorem on the existence of a unimodular matrix with prescribed first row or column. Putting also $y_1 = x_1$, this takes the given form into $a_{12}x_1x_2+\psi'$, where $\psi' = \psi'(x_2,\ldots,x_k)$ is the transform of ψ. Gathering together all the terms involving x_2, we see that this form is of the type (10), with $g = \psi'(0,x_3,\ldots,x_k)$.

Now let s be fixed, and satisfy $0 \leqslant s \leqslant 8$; we apply Lemma 4 to prove the theorem by induction from $k-2$ to k. The known cases of the theorem mentioned in § 2 give us a starting-point for each of the nine inductions; and also show that we may suppose $k(f) \geqslant 3$, f a zero form, and so use Lemma 4. We see from (10) that $s(g) = s(f)$, and so the inductive hypothesis gives

$$(\min^+ g)^{k-2} \leqslant c(g)|d(g)|, \tag{11}$$

where $c(g) = c(f) = 1, \frac{1}{2}, \frac{1}{3}$, or $\frac{1}{4}$ is the constant on the right of (7), depending only on $s(g) = s(f)$ modulo 8.

Now from (11) we see that f represents every integer $a_{12}x_1+a_{22}$, and clearly some such integer is positive and does not exceed a_{12}, so that

$$\min^+ f \leqslant a_{12}. \tag{12}$$

From (11), (12) we have

$$(\min^+ f)^k \leqslant a_{12}^2 c(f)|d(g)|$$

since by putting $x_2 = 0$ in (11) we see that f represents $\min^+ g$. Now from (1), (11) it is easily seen that $d(f) = a_{12}^2 d(g)$, and so we have

$$(\min^+ f)^k \leqslant c(f)|d(f)|,$$

which is (7).

For equality in (7) we have to have equality in (11) and in (12), and also $\min^+ g = a_{12}$, which by the inductive hypothesis implies that g is identically congruent to 0 modulo a_{12}. Hence from (11), with $x_2 = 1$, we see that f represents every integer congruent, for any choice of $x_3,\ldots, x_k$, to $a_{22}+a_{32}x_3+\ldots+a_{k2}x_k$ modulo a_{12}. There can clearly be equality in (12) only if every such integer is a multiple of a_{12}, in which case a_{12} is a divisor of f, whence $a_{12} = 1$ by the assumption that f is primitive. But this gives $\min^+ f = 1$, and now equality in (7) implies (4).

By the remark at the end of § 3, the theorem is now proved except in the three cases k, $s = 3, -1$; $4, -2$; $5, -1$.

5. In the last of these three cases, if we either assume the theorem to be true in the first, or use the result of (4), we see that g in (10), which has $k(g) = 3$, $s(g) = -1$, satisfies

$$(\min^+ g)^3 \leqslant \tfrac{27}{16}|d(g)|.$$

With an obvious modification of the argument of § 4, this will give (7): that is,

$$(\min^+ f)^5 \leqslant |d(f)|,$$

with equality only if the left-hand member is 1, if we can replace (12) by

$$\min^+ f \leqslant \max(1, \tfrac{3}{4}a_{12}). \tag{13}$$

We may moreover assume that g is also a zero form; since otherwise, as noted in § 2, (7) holds for g with strict inequality, and the argument of § 4 works. So applying Lemma 4 to g we may suppose (changing, possibly, the values of a_{32}, a_{42}, a_{52}) that

$$f = x_2(a_{12}x_1+a_{22}x_2+a_{32}x_3+a_{42}x_4+a_{52}x_5)+ \\ +x_4(a_{34}x_3+a_{44}x_4+a_{54}x_5)+a_{55}x_5^2, \tag{14}$$

with $a_{55} < 0$ because $s(f) = -1$.

The argument used to prove Lemma 4 shows (on permuting the suffixes and noting that the coefficient of x_3^2 in (14) is zero) that f is equivalent to a form of the type (10) with a'_{12}, the g.c.d. of a_{32}, a_{34}, in place of a_{12}. Suppose temporarily that, if there is any choice, we take the least possible a_{12}: we thus have a_{32} and a_{43} both divisible by a_{12}; otherwise we should have $a'_{12} < a_{12}$. Hence, by putting $x_1 - a_{12}^{-1}a_{32}x_3$ for x_1, we may clearly suppose that $a_{32} = 0$. If now a_{12} is a proper divisor of a_{34}, then (12) gives $\min^+ f \leqslant \frac{1}{2}a_{34}$; so (13) holds after permuting the suffixes so as to interchange a_{12}, a_{34}. We may therefore suppose that $a_{12} = a_{34}$, $a_{32} = 0$.

Now in (14) we may clearly suppose that $0 < a_{22} \leqslant a_{12}$; and, since f represents a_{22}, we may suppose that $\frac{3}{4}a_{12} < a_{22} \leqslant a_{12}$; otherwise (13) holds. Similarly $\frac{3}{4}a_{12} < a_{44} \leqslant a_{12} = a_{34}$, and trivially

$$0 \leqslant a_{42} \leqslant \tfrac{1}{2}a_{12}.$$

But now (14) shows that

$$0 < f(0, 1, 1, -1, 0) \leqslant \tfrac{3}{4}a_{12},$$

whence (13).

6. It remains to consider the two cases $k, s = 3, -1;\ 4, -2$. In each of these we may assume (10), for the reason noted in § 5, and in either case we write

$$f_0 = f(0, x_2, \ldots, x_k).$$

It is easily seen that by a suitable substitution $x_1 \to x_1 + hx_2$ we may suppose that

$$0 < d(f_0) \leqslant 4a_{12}|a_{33}| \quad (\text{if } k = 3), \tag{15}$$

$$0 > d(f_0) \geqslant a_{12}\,d(g) \quad (k = 4), \tag{16}$$

where in (15) we write $a_{33}x_3^2$ ($a_{33} < 0$) for the 1-ary form g, while, in (16), $d(g) < 0$ since g is negative definite. Then (16) gives us that f_0 is

a ternary form with signature -1, or else a positive-definite form; but, because of the negative section g of f_0, the latter alternative is impossible.

Hence in case (15) we have (by the binary case of the theorem)

$$(\min^+ f)^2 \leqslant (\min^+ f_0)^2 \leqslant 4a_{12}|a_{33}| = 4a_{12}^{-1}|d(f)|.$$

And this will give (7), with strict inequality, if we have $\min^+ f < \frac{1}{4}a_{12}$.

In case (16) we have

$$(\min^+ f)^3 \leqslant \tfrac{27}{16}a_{12}|d(g)| = \tfrac{27}{16}a_{12}^{-1}|d(f)|, \tag{17}$$

by the case $k, s = 3, -1$ of the theorem or by the weaker result noted in § 5. This gives

$$(\min^+ f)^4 < \tfrac{1}{3}|d(f)|,$$

which is (7) with strict inequality, if we have $\min^+ f < \frac{16}{81}a_{12}$. It is, however, easy to deal separately with the case in which f_0 is a multiple of a form equivalent to one of (5). Excluding this case, we may omit the factor $\frac{27}{16}$ in (17), and we have (7) with strict inequality if

$$\min^+ f < \tfrac{1}{3}a_{12}.$$

7. It will now be clear that the two remaining cases will be disposed of, after a finite calculation, if we can prove that in each of them we have $\min^+ f < \frac{1}{4}a_{12}$ except for finitely many values of a_{12}. Since the proof of no other case depends on either of these two, I shall merely sketch the proof that this is so, omitting the calculations.

Let c be the product of the distinct odd prime factors of a_{12} in (10), and write $\delta = 0, 1, 2, 3$ according as

$$a_{12} \text{ is odd}, \quad a_{12} \equiv 2 \pmod 4, \quad a_{12} \equiv 4 \pmod 8, \quad a_{12} \equiv 0 \pmod 8.$$

Then we have the lemma:

LEMMA 5. *There exists an integer n_0 prime to $2c$ such that (for $k \geqslant 4$, f indefinite, or $k = 3$, $s = -1$) f represents every positive integer n satisfying*

$$n \equiv n_0 \pmod{2^\delta}, \qquad (n \mid p) = (n_0 \mid p) \text{ for } p \text{ odd}, p \mid c \tag{18}$$

(where $(n \mid p)$ is the Legendre symbol).

Proof. We choose n_0, if possible, to be an integer prime to $2c$ and representable by f with $x_1 = 0$, $x_2 = 1$: say $n_0 = f(0, 1, t_3, \ldots, t_k)$. This choice is easily seen to be possible if the coefficients $a_{32}, \ldots, a_{k2}$ are all even. If one of these coefficients is odd, it may be impossible to make $f(0, 1, t_3, \ldots, t_k)$ odd, but we can make it prime to c and it is easy to prove that in this case f is capable of every odd residue modulo 8; we choose n_0 as before modulo c, and odd, and otherwise arbitrary.

Since (10) shows that f represents every integer $a_{12}x_1 + a_{22}$, (8) is

soluble for all n if m is a prime power p^r with $p \nmid a_{12}$: that is, $p \nmid 2c$, or $p \nmid c$ if a_{12} is odd. If in (8) we have $m = p^r$ with p odd and $p \mid c$, then $(18)_2$ implies the solubility of (8). If we have $m = 2^r$, then $(18)_1$, with the first choice of n_0, ensures that

$$f(x_1, 1, t_3, \ldots, t_k) \equiv n \pmod{2^r}$$

is soluble for x_1. If the alternative choice of n_0 is necessary, then $(18)_1$ makes n odd (if $2 \mid a_{12}$) and so ensures the solubility of (8).

Hence (8) is soluble in all cases, and Lemma 1 shows that f represents n.

It is clear that (18) can be satisfied with $0 < n < 2^\delta c$, and this gives $\min^+ f < 2^\delta c$. If $2^\delta c \leqslant \frac{1}{4}a_{12}$, we have the desired result $\min^+ f < \frac{1}{4}a_{12}$; and this also follows if we have $\min^+ f < 2^{\delta-2}c$. It suffices to prove this last inequality for large c: that is for c with a large prime factor p (above some numerical bound).

The problem reduces easily to showing that for sufficiently large p (in fact for $p \geqslant 29$) there is a quadratic residue and a quadratic non-residue modulo p in every open interval of length $\frac{1}{4}p$. To prove this for residues, consider the squares between 0 and p. Then it follows for non-residues if $p \equiv -1 \pmod 4$.

To verify the result for non-residues when $p \equiv 1 \pmod 4$, consider the integers $\pm 2z^2$ or $\pm 3z^2$ between $\pm\frac{1}{2}p$, if either 2 or 3 is a non-residue. If not, assume for the moment that there is a non-residue between 1 and $\frac{1}{8}p$, let $q > \frac{1}{16}p$ be the greatest such non-residue, and consider the numbers (all non-residues) $\pm q$, $\pm 2q$, $\pm 4q$, $\pm 6q$, $\pm 8q$.

Finally, if no such q exists, then, for $u = 1, 2, 3, 4$, all the integers $\pm n$ with n in the interval $(\frac{1}{8}(u-1)p, \frac{1}{8}up)$ and divisible by u are residues. They are all incongruent modulo p, and a simple calculation shows that there are more than $\frac{1}{2}(p-1)$ of them. This contradiction completes the proof.

8. With integral, primitive forms considerations of homogeneity do not apply, and there is no reason why the exponent γ in an estimate $\min^+ f = O(|d|^\gamma)$ should be k^{-1}. I have shown (**10**) that

$$\min^+ f = O\big(|d|^{\epsilon+1/(2k-2)}\big).$$

From this, it should be possible to deduce (7) for $|d|$ exceeding some bound d_0. It would, however, be impracticable to obtain the present result in this way since d_0 would be enormously large.

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ON PHASE-INVARIANT AND GAUGE-INVARIANT SPINOR ANALYSIS

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1. In the gauge-covariant formulation of spinor analysis of Infeld and van der Waerden (**1**)† the metric spinor $\gamma_{\mu\nu}$ is subject to the condition

$$(\gamma_{\mu\dot{\nu}}\gamma_{\alpha\beta})_{;\varepsilon} = 0. \tag{1}$$

A gauge-transformation is understood to be a spin transformation

$$\Lambda^{\nu}_{\mu} = \delta^{\nu}_{\mu} e^{i\phi/2}, \tag{2}$$

for under this a contravariant spin vector a^{λ} transforms according to

$$a'^{\lambda} = e^{i\phi/2} a^{\lambda}, \tag{3}$$

whilst at the same time the imaginary part Φ_s of $\Gamma^{\alpha}_{\alpha s}$ (which is arbitrary) transforms according to

$$\Phi'_s = \Phi_s - \phi_{,s}. \tag{4}$$

Any tensor-spinor equation is therefore necessarily gauge-covariant in this version of spinor analysis. The disadvantage of (1) is that the covariant derivative of $\gamma_{\mu\nu}$ does not vanish in general; so that the operations of raising and lowering spinor indices do not commute with the operation of forming the covariant derivative. It is therefore enticing to consider the possibility of replacing (1) by the stronger condition

$$\gamma_{\mu\nu;s} = 0, \tag{5}$$

but, as pointed out by Bade and Jehle (**2**), the vector Φ_s would thereby be eliminated from the theory and its gauge-covariant character thus lost.

In the first place, then, it may be remarked that it is possible to retain the advantages of both gauge-covariance and the formal covariant constancy of $\gamma_{\mu\nu}$ by a minor modification of the definition of covariant differentiation. Accordingly, let subscripts following a semicolon now denote *gauge-invariant covariant differentiation*. Then, if $Z^{\nu\dots}_{\mu\dots k\dots}$ is a typical tensor-spinor,

$$Z^{\nu\dots}_{\mu\dots k\dots;s} = Z^{\nu\dots}_{\mu\dots k\dots,s} + \Gamma^{\nu}_{\alpha s} Z^{\alpha\dots}_{\mu\dots k\dots} + \dots - \Gamma^{\lambda}_{\mu s} Z^{\nu}_{\alpha\dots k\dots} - \dots$$
$$- \Gamma^{l}_{ks} Z^{\nu\dots}_{\mu\dots l\dots} - \dots - i\omega\kappa_s Z^{\nu\dots}_{\mu\dots k\dots}. \tag{6}$$

Here κ_s is a given real (spin-invariant) vector, and ω is the gauge-weight

† Their notation is used here, except where otherwise indicated.

of $Z^{\nu\ldots}_{\mu\ldots\kappa\ldots}$: the gauge-weight of a spinor is said to be ω if it multiplies itself by the factor $e^{i\omega\phi}$ under the transformation (2). Here (6) differs from the usual definition only through the presence of the last term on the right. If, as usual, one writes $\gamma_{12} = \sqrt{\gamma}\, e^{i\theta}$, the electromagnetic potential Φ_σ is then to be taken (apart from a numerical factor) as

$$\Phi_\sigma = \kappa_\sigma + \theta_{,\sigma}. \tag{7}$$

2. (*a*) In the remainder of this paper the term 'gauge-covariance' as used above will be replaced by the term *phase-covariance*, as suggested by Weyl (**3**), whilst the former will be used only in its original sense of Weyl's theory (**4**). Then (6) is very reminiscent of the definition of gauge-invariant covariant differentiation in a Weyl space [cf. Buchdahl (**5**)]. An important difference, however, is that in (6) the $\Gamma^\beta_{\alpha\sigma}$ themselves are not phase-invariant, whilst the components of the linear connexion of a W_n are gauge-invariant. This difference in behaviour arises from the fact that, whereas phase transformations as defined above are elements of the group of allowed spin transformations, gauge transformations are not in general elements of the group of allowed coordinate transformations. The correspondence between the two kinds of transformations can accordingly be made closer by a redefinition of phase transformation: the latter is now to be understood to mean the transformation (2) coupled with the simultaneous transformation

$$\kappa_\sigma \to \kappa_\sigma + \phi_{,\sigma}, \tag{8}$$

equation (6), along with the definition of ω, being retained formally unaltered. In a spin transformation alone κ_σ is invariant. If $\overset{0}{\Gamma}{}^\nu_{\mu\sigma}$ are the components of the linear spinor connexion when $\kappa_\sigma = 0$ (and phase transformations are not contemplated), then

$$\Gamma^\nu_{\mu\sigma} = \overset{0}{\Gamma}{}^\nu_{\mu\sigma} + \tfrac{1}{2} i \delta^\nu_\mu \kappa_\sigma. \tag{9}$$

The right-hand member of (9) is obviously invariant under phase transformations since under (2) alone

$$\overset{0}{\Gamma}{}^\nu_{\mu\sigma} \to \overset{0}{\Gamma}{}^\nu_{\mu\sigma} - \tfrac{1}{2} i \delta^\nu_\mu \phi_{,\sigma}.$$

(*b*) A further change in emphasis towards the formalism of gauge-invariance is achieved by avoiding any reference to the spin transformation (2) in the definition of phase transformation. Instead, the latter is now defined as the simultaneous transformations

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu} e^{-i\phi}, \qquad \kappa_\sigma \to \kappa_\sigma + \phi_{,\sigma}. \tag{10}$$

The phase-weight is defined by the convention that a spinor Z (indices suppressed) has phase-weight ω if under the transformation (10) it multiplies itself by $e^{i\omega\phi}$. The phase-weight of $\sigma^{k\dot{\mu}\nu}$ is zero,† and the weight of any other spinor (whose weight is not implied by the weights of the basic spinors) must be prescribed *a priori* whenever it is not dictated by considerations of consistency. This means that under a phase transformation (10) the transformation (3) of a contravariant spin vector is not automatically implied: that is, it will be so implied only after a^λ has been *prescribed* to have the phase-weight $+\frac{1}{2}$.

3. The spinor analysis associated with a Riemann space V_4 is characterized by the equations

$$\sigma^{k\dot{\mu}\nu}\sigma^{l}_{\dot{\mu}\nu} = g^{kl}, \quad \sigma^{k\dot{\mu}\nu}\sigma_{k}^{\dot{\alpha}\beta} = \gamma^{\dot{\mu}\dot{\alpha}}\gamma^{\nu\beta}, \quad \sigma^{k\dot{\mu}\nu}{}_{:s} = 0, \quad \gamma_{\mu\nu:s} = 0, \quad (11.1\text{–}4)$$

phase-covariance being temporarily abandoned. (Indices following a colon denote covariant differentiation in V_4.) The components of the linear connexions shall be $\underset{0}{\Gamma}{}^{l}_{sk}$, $\underset{0}{\Gamma}{}^{\nu}_{\mu k}$ (the former, of course, being Christoffel three-index symbols). Let W_4 be the four-dimensional Weyl space whose fundamental tensor (for a chosen gauge) is the metric tensor of the V_4 above, whilst its fundamental vector is k_s. Let there be associated with the W_4 basic spinors $\sigma^{k\dot{\mu}\nu}$, $\gamma_{\alpha\beta}$ which obey equations (11.1) and (11.2). Then joint gauge and phase transformations are defined by

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu}e^{-if-i\phi}, \qquad k_s \to k_s + f_{,s}, \qquad \kappa_s \to \kappa_s + \phi_{,s}, \qquad (12)$$

where f and ϕ are arbitrary real scalars, whilst k_s and κ_s are arbitrary real spin-invariant vectors. A spinor then has gauge-weight v and phase-weight ω if under the transformation (12) it takes a factor $e^{vf+i\omega\phi}$. It is convenient to prescribe the gauge-weight of $\sigma^{k\dot{\mu}\nu}$ as zero, whereby g_{ij} acquires unit weight. The gauge-invariant phase-invariant covariant derivative of a typical tensor-spinor $Z^{\nu\ldots}_{\dot{\mu}\ldots k\ldots}$ is then defined as

$$Z^{\nu\ldots}_{\dot{\mu}\ldots k\ldots;s} = Z^{\nu\ldots}_{\dot{\mu}\ldots k\ldots,s} + \Gamma^{\nu}_{\lambda s}Z^{\lambda\ldots}_{\dot{\mu}\ldots k\ldots} + \ldots - \Gamma^{\dot{\lambda}}_{\dot{\mu}s}Z^{\nu\ldots}_{\dot{\lambda}\ldots k\ldots} - \ldots$$
$$- \Gamma^{l}_{ks}Z^{\nu\ldots}_{\dot{\mu}\ldots l\ldots} - \ldots - (vk_s + i\omega\kappa_s)Z^{\nu\ldots}_{\dot{\mu}\ldots k\ldots}, \qquad (13)$$

whilst in place of (11.3, 4) one now has

$$\sigma^{k\dot{\mu}\nu}{}_{;s} = 0, \qquad \gamma_{\mu\nu;s} = 0. \qquad (14.1, 2)$$

Setting for any component of either linear connexion Γ (indices suppressed)

$$\Delta = \Gamma - \underset{0}{\Gamma}, \qquad (15)$$

† Evidently any hermitian spinor must have zero weight.

one has, from (11.3, 4) and (14.1, 2), since for the chosen gauge and phase $\sigma^{k\dot{\mu}\nu}$ and $\gamma_{\mu\nu}$ are the same for V_4 and W_4,

$$-\Delta^k_{st}\sigma^{l\dot{\mu}\nu}+\Delta^\mu_{\dot{\lambda}s}\sigma^{k\dot{\lambda}\nu}+\Delta^\nu_{\lambda s}\sigma^{k\dot{\mu}\lambda}=0, \tag{16}$$

$$-\Delta^\lambda_{\lambda s}+(\tfrac{1}{2}k_s+ik_s)=0. \tag{17}$$

Now Δ^l_{st} is given by (**5**) (1.1), viz.

$$\Delta^l_{st}=-\tfrac{1}{2}(\delta^l_s k_t+\delta^l_t k_s-g_{st}g^{lr}k_r). \tag{18}$$

Then it is easily verified that the solution of equations (16) and (17) is

$$\Delta^\nu_{\mu s}=\tfrac{1}{2}(\sigma_{s\dot{\lambda}\mu}\sigma^{l\dot{\lambda}\nu}k_l+i\delta^\nu_\mu\kappa_s). \tag{19}$$

Thus (19) provides the explicit form of the gauge-invariant and phase-invariant spinor connexion. From it one may form the gauge-invariant and phase-invariant spin curvature tensor†

$$P^\nu{}_{\mu st}=2\Gamma^\nu_{\mu[t,s]}+2\Gamma^\lambda_{\mu[t}\Gamma^\nu_{\lambda s]}. \tag{20}$$

The Ricci's identity reads, in the case of a contravariant spin vector for example,

$$-2\eta^\nu{}_{;[st]}=P^\nu{}_{\mu st}\eta^\mu+(vf_{st}+i\omega\phi_{st})\eta^\nu, \tag{21}$$

where

$$f_{st}=2k_{[s,t]}, \qquad \phi_{st}=2\kappa_{[s,t]}. \tag{22}$$

Applying this to the identity $\gamma_{\mu\nu;[st]}=0$ one has at once

$$P^\lambda{}_{\lambda st}=-(\tfrac{1}{2}f_{st}+i\phi_{st}). \tag{23}$$

Using (19) in (20) one has after some manipulation the detailed result

$$P^\nu{}_{\mu st}=\overset{0}{P}{}^\nu{}_{\mu st}+K_{r[s}\sigma_{t]\dot{\lambda}\mu}\sigma^{r\dot{\lambda}\nu}-\tfrac{1}{2}i\delta^\nu_\mu\phi_{st}, \tag{24}$$

where

$$K_{st}=k_{s:t}+\tfrac{1}{2}k_s k_t-\tfrac{1}{4}g_{st}k_l k^l. \tag{25}$$

The identity

$$\sigma^{k\dot{\mu}\nu}{}_{;[st]}=0$$

gives the relation between $P^\nu{}_{\mu st}$ and the curvature tensor of W_4, viz.

$$R^k{}_{lst}\sigma^{l\dot{\mu}\nu}+P^{\dot{\mu}}{}_{\dot{\lambda}st}\sigma^{k\dot{\lambda}\nu}+P^\nu{}_{\lambda st}\sigma^{k\dot{\mu}\lambda}=0 \tag{26}$$

since $v=\omega=0$ here. The identity (26) is formally identical with the corresponding relation in a V_4.

In conclusion it may be mentioned that the considerations above can also be translated into the language of the ϵ-formalism [cf. (**1**)], but the analysis is then somewhat overburdened by the multiplicity of weights which characterize the general spinor, viz. its coordinate-weight, spin-weight and spin-antiweight, gauge-weight and phase-weight. The use

† Brackets enclosing indices act in every case only on one type of index, e.g. here only on the *tensor* indices s and t.

of the γ-formalism at least prevents the necessary appearance of two of these.

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SOME REMARKS ON TAUBERIAN THEOREMS

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1. Introduction

In § 5 of this paper we are interested in tauberian theorems for the Laplace and Abel transforms. The tauberian conditions in that section are of one of the two forms

$$\liminf_{x\to\infty} \inf_{x<y\leqslant\lambda x} \{s(y)-Ms(x)\} \geqslant -c$$

for some $\lambda > 1$, $M > 0$, and finite c. (1.1)

$$\lim_{\lambda\downarrow 1} \liminf_{x\to\infty} \inf_{x<y\leqslant\lambda x} \{s(y)-Ms(x)\} \geqslant -c$$

for some $M > 0$ and finite c. (1.2)

The tauberian condition (1.1) with $M = 1$ was treated by T. Vijayaraghavan [see (**16**)]; (1.1) with $c = 0$ includes R. C. Buck's condition for functions of 'bounded decrease' $s(y)-Ms(x) \geqslant 0$ for $0 < x \leqslant y \leqslant \lambda x$ [see (**3**)]; (1.2) for $M = 1$ and $c = 0$ is R. Schmidt's well-known tauberian condition.

In § 5 we are interested in tauberian theorems with tauberian conditions of the forms

$$\liminf_{x\to\infty} \{s(\lambda x)-s(x)\} > -\infty \quad (x \uparrow \infty) \tag{1.3}$$

for some $\lambda > 1$,

$$\liminf_{x\to\infty} \{s(\lambda x)-s(x)\} > -\infty \quad (x \uparrow \infty) \tag{1.4}$$

for each $\lambda > 1$ of a set of numbers $\lambda > 1$ such that the set of numbers $\log\lambda$ is of positive Lebesgue measure. It might be remarked here that the condition

$$\liminf_{x\to\infty} \{s(\lambda x)-s(x)\} > -K > -\infty$$

uniformly in λ satisfying $1 < \lambda \leqslant \lambda_0$ is the same as (1.1) with $M = 1$ and $\lambda = \lambda_0$.

In § 3 we investigate the inclusion relations between (L, α) methods of summability of different orders. (For the definition of the (L, α) methods see § 2.)

In § 2 we are interested in the 'iteration product problem' of the (L, α) and continuous Hausdorff transformations. The 'iteration product problem' might be stated thus. Suppose A and B regular transformations of a sequence $\{s_n\}$ or a function $s(x)$. Denote by AB the

iteration product which associates with a given sequence (or function) the A-transform of its B-transform. For which pairs A, B of transformations does A-summability imply AB-summability? This problem for some pairs A, B was treated by me in (7) and (8), by O. Szász in (13) and (14), by C. T. Rajagopal and me in (9), by C. T. Rajagopal in (11), and by M. S. Ramanujan in (12).

In § 4 we define a class of transformations, the $M(\alpha, \beta)$ transformations, and give some of their properties. We use these transformations in § 6.

I wish to thank the referee for some interesting remarks.

2. The iteration product of Hausdorff and (L, α) transformations

In this section we suppose that the function $s(x)$ is bounded and B-measurable in every finite interval of $x \geqslant 0$. Every integral over a finite interval will be a Lebesgue–Stieltjes integral. By $\int^{\infty-} f(x)\,dx$ we denote

$$\lim_{c \uparrow \infty} \int^{c} f(x)\,dx$$

if the last limit exists. Only when $\int^{\infty-} |f(x)|\,dx$ exists we denote $\int^{\infty-} f(x)\,dx$ by $\int^{\infty} f(x)\,dx$.

Definition. A function $s(x)$ is summable (L, α) to s for some fixed $\alpha > -1$ if

$$L^{(\alpha)}(\tau) \equiv L^{(\alpha)}\{\tau, s(x)\} \equiv \frac{\tau^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^{\infty-} s(t) t^{\alpha} e^{-\tau t}\,dt$$

exists for all $\tau > 0$ and $\lim_{\tau \downarrow 0} L^{(\alpha)}(\tau) = s$.

It is easy to see that the (L, α) transformations are regular for $\alpha > -1$; that is, from

$$\lim_{x \to \infty} s(x) = s$$

follows

$$\lim_{\tau \downarrow 0} L^{(\alpha)}(\tau) = s$$

for each $\alpha > -1$.

Let $\phi(x)$ be of bounded variation in $0 \leqslant x \leqslant 1$. The function $t(x)$ defined by

$$t(x) \equiv \int_0^1 s(xt)\,d\phi(t) \quad (x \geqslant 0)$$

is called the '(continuous) Hausdorff transform of $s(x)$ by means of $\phi(x)$', or, in short, an $[H, \phi(x)]$ or H transform of $s(x)$. If $\phi(x)$ is absolutely continuous in $0 \leqslant x \leqslant 1$, we suppose in the definition of $t(x)$ that $s(x)$

is L-measurable in each finite interval of $x \geqslant 0$ and not necessarily Borel measurable. We say that $s(x)$ is summable $[H, \phi(x)]$ to s if $\lim_{x \to \infty} t(x) = s$.

Again $\sigma^{(\alpha)}(x)$, the Cesàro transform of order $\alpha \geqslant 0$ of $s(x)$ (in short, the (C, α) transform) is defined for $\alpha > 0$ as the $[H, 1-(1-x)^{\alpha}]$ transform of $s(x)$ and for $\alpha = 0$ as equal to $s(x)$. It is known that the (C, α) transformations are regular for $\alpha \geqslant 0$.

It is known [see (4)] that an $[H, \phi(x)]$ transformation is regular if and only if

$$\phi(0) = \phi(0+), \qquad \phi(1)-\phi(0) = 1.$$

We shall suppose (as we may) to the end of this paper that for regular $[H, \phi(x)]$ transformations we have $\phi(0) = 0$ and $\phi(1) = 1$.

The main result of this section is

THEOREM 2.1. *Let $s(x)$ be summable (L, α) to s for some $\alpha > -1$. If $t(x)$ is any regular $[H, \phi(x)]$ transform of $s(x)$, then for all positive values of w*

$$\frac{w^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^{\infty -} t(v) v^{\alpha} e^{-v/w}\, dv = \int_0^1 \left\{ \frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^{\infty -} s(u) u^{\alpha} e^{-u/wx}\, du \right\} d\phi(x),$$

and further $t(x)$ is summable (L, α) to s.

In order to justify the interest in the theorem for non-absolutely convergent integrals we may take the function

$$s(x) = \exp(e^t+t)\sin \exp e^t,$$

which is summable $(L, 1)$ to zero and its $(L, 1)$ transform is not absolutely convergent. [For this example, see (17) 46.]

Theorem 2.1 for absolutely convergent and uniformly bounded $L^{(\alpha)}\{\tau, s(x)\}$ transforms for $0 < \tau < \infty$ is included in Theorem II of Rajagopal's paper (11).

We need two lemmas.

LEMMA 2.1. *Let $L^{(\alpha)}\{\tau, s(x)\}$ exist for some $\alpha > -1$ and for all $\tau > 0$. If $s(0+)$ exists, then $\lim_{\tau \uparrow \infty} L^{(\alpha)}(\tau) = s(0+)$.*

Proof. For $x \geqslant 0$ define

$$\alpha(x) = \int_0^x s(t) t^{\alpha}\, dt.$$

Then

$$f(\tau) = \int_0^{\infty -} e^{-\tau t}\, d\alpha(t) = \int_0^{\infty -} s(t) t^{\alpha} e^{-\tau t}\, dt.$$

Now, as $x \downarrow 0$, $\alpha(x) \sim \int_0^x s(0+)t^\alpha\,dt = \dfrac{s(0+)}{\alpha+1}x^{\alpha+1}$.

Write $\gamma = \alpha+1, \quad A = \Gamma(\alpha+1)s(0+)$.

Then [(17) 181, Theorem 1 (3)] as $\tau \downarrow 0$,

$$f(\tau) \sim \Gamma(\alpha+1)\tau^{-(\alpha+1)}s(0+),$$

which proves the lemma. Incidentally this theorem is due to Hardy and Littlewood [see (5) 27, Lemma β].

LEMMA 2.2. *Let $L^{(\alpha)}\{\tau, s(x)\}$ exist for some $\alpha > -1$ and for all $\tau > 0$. Suppose that $s(0+)$ exists and define $L^{(\alpha)}\{+\infty, s(x)\} = s(0+)$. Then $L^{(\alpha)}\{\tau, s(x)\}$ is continuous in τ for $0 < \tau \leqslant +\infty$.*

The proof follows from Lemma 2.1 and the known fact that a Laplace transform is a regular analytic function inside its region of convergence.

Proof of Theorem 2.1. We may suppose, as we shall do in this proof, that $s(x) = 0$ for $0 \leqslant x \leqslant 1$. We see this by the following two facts. Define $\bar{s}(x)$ by

$$\bar{s}(x) = \begin{cases} s(x) & (0 \leqslant x \leqslant 1), \\ 0 & (x > 1). \end{cases}$$

Then $\bar{l}(x)$ is defined as the $[H, \phi(x)]$ transform of $\bar{s}(x)$. First, by Fubini's theorem, we have, for $w > 0$, since $s(x)$ is bounded in $0 \leqslant x \leqslant 1$,

$$\begin{aligned} \int_0^1 \left\{ \frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^1 s(u)u^\alpha e^{-u/wx}\,du \right\} d\phi(x) \\ &= \int_0^1 \left\{ \frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^\infty \bar{s}(u)u^\alpha e^{-u/wx}\,du \right\} d\phi(x) \\ &= \frac{w^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^1 \left\{ \int_0^\infty \bar{s}(xv)v^\alpha e^{-v/w}\,dv \right\} d\phi(x) \\ &= \frac{w^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^\infty \bar{l}(v)v^\alpha e^{-v/w}\,dv. \end{aligned}$$

Secondly, $\lim_{x \downarrow 0} \dfrac{x^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^1 s(t)t^\alpha e^{-xt}\,dt = 0.$

By Lemma 2.2 and $\lim_{x \downarrow 0} L^{(\alpha)}(x) = s$ it follows that, for each constant w satisfying $0 < w < +\infty$, $L^{(\alpha)}(wx)$ is continuous for $0 \leqslant x \leqslant +\infty$;

therefore this function is bounded for $0 \leqslant x \leqslant +\infty$ and the repeated integral

$$\int_0^1 L^{(\alpha)}\left(\frac{1}{wx}\right) d\phi(x)$$

exists. First we show that the last repeated integral is equal to

$$\lim_{A \uparrow \infty} \int_0^1 \left\{ \frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_1^{Ax} s(u) u^{\alpha} e^{-u/xw}\, du \right\} d\phi(x).$$

It is easy to show [see (17) 39 Theorem 2.2a for a similar result] that, if $L^{(\alpha)}(\tau_0)$ exists for some $\alpha > -1$ and $\tau_0 > 0$, then

$$s_1(x) \equiv \int_0^x s(t)\, dt = O(x^{-\alpha} e^{\tau_0 x}) \quad \text{as } x \to \infty.$$

From the last remark we obtain on integrating by parts

$$(wx)^{-(\alpha+1)} \int_{Ax}^{\infty -} s(u) u^{\alpha} e^{-u/wx}\, du = -(Ax)^{-1} s_1(Ax) A^{\alpha+1} w^{-(\alpha+1)} e^{-A/w} -$$

$$-(wx)^{-(\alpha+1)} \int_{Ax}^{\infty} s_1(u)\left(\alpha - \frac{u}{xw}\right) u^{\alpha-1} e^{-u/wx}\, du. \tag{2.1}$$

For x satisfying $0 \leqslant x \leqslant \delta/A$, for some fixed $\delta > 0$ and for $A > \delta$, we have

$$\left| \frac{s_1(Ax)}{Ax} A^{\alpha+1} w^{-(\alpha+1)} e^{-A/w} \right| \leqslant C\left(\frac{A}{w}\right)^{\alpha+1} e^{-A/w},$$

where $|\sigma^{(1)}(y)| \leqslant C \equiv C(\delta)$ for $0 \leqslant y \leqslant \delta$. Accordingly

$$\frac{s_1(Ax)}{Ax} A^{\alpha+1} w^{-(\alpha+1)} e^{-A/w} \to 0, \tag{2.2}$$

as $A \to +\infty$ uniformly in x, where $0 \leqslant x \leqslant \delta/A$. For $\delta/A \leqslant x \leqslant 1$ we have, if we choose a γ such that $w\gamma < \frac{1}{2}$,

$$\left| \frac{s_1(Ax)}{Ax} \left(\frac{A}{w}\right)^{\alpha+1} e^{-A/w} \right| \leqslant K\delta^{-(\alpha+1)} \left(\frac{A}{w}\right)^{\alpha+1} \exp\{-(w^{-1}-\gamma)A\},$$

where $|\sigma^{(1)}(u)| \leqslant K\delta^{-(\alpha+1)} e^{\gamma A}$ $(u \geqslant \delta)$. Thus

$$\frac{s_1(Ax)}{Ax} \left(\frac{A}{w}\right)^{\alpha+1} e^{-A/w} \to 0 \tag{2.3}$$

as $A \to \infty$, uniformly in x $(\delta/A \leqslant x \leqslant 1)$. Similarly for $\delta/A \leqslant x \leqslant 1$, if A is sufficiently large, $\gamma w < \frac{1}{2}$ and $A/w > \alpha$,

$$\left|(wx)^{-(\alpha+1)} \int_{Ax}^{\infty} s_1(u) u^{\alpha-1} e^{-u/wx} \left(\alpha - \frac{u}{wx}\right) du\right|$$

$$\leqslant (wx)^{-(\alpha+1)} \int_{Ax}^{\infty} |s_1(u)| u^{\alpha} (wx)^{-1} e^{-u/wx}\, du$$

$$\leqslant K(wx)^{-(\alpha+1)} \int_{Ax}^{\infty} (wx)^{-1} \exp\left\{-\left(\frac{1}{wx} - \gamma\right)u\right\} du$$

$$= K(wx)^{-(\alpha+1)}(1-\gamma wx)^{-1} \exp\{-(w^{-1}-\gamma x)A\}$$

$$\to 0 \text{ as } A \to \infty, \quad \text{uniformly in } x\ (\delta/A \leqslant x \leqslant 1). \quad (2.4)$$

For $0 \leqslant x \leqslant \delta/A < 1$ we have, if A is sufficiently large and $A/w > \alpha$,

$$\left|(wx)^{-(\alpha+1)} \int_{Ax}^{\infty} s_1(u)\left(\alpha - \frac{u}{wx}\right) u^{\alpha-1} e^{-u/wx}\, du\right|$$

$$\leqslant (wx)^{-(\alpha+2)} \int_{Ax}^{\infty} |\sigma^{(1)}(u)| u^{\alpha+1} e^{-u/wx}\, du.$$

Now, if, in addition, $0 < \gamma < 1$, $\gamma w < 1$, and $A > 1$, we get, since, for $0 \leqslant u \leqslant 1$, $|\sigma^{(1)}(u)|$ is bounded and, for $1 \leqslant u < +\infty$, $\sigma^{(1)}(u) = O(e^{\gamma u})$,

$$\left|(wx)^{-(\alpha+1)} \int_{Ax}^{\infty} s_1(u)\left(\alpha - \frac{u}{wx}\right) u^{\alpha-1} e^{-u/wx}\, du\right|$$

$$= O\left[(wx)^{-(\alpha+2)} \int_{Ax}^{\infty} u^{\alpha+1} \exp\left\{-\left(\frac{1}{wx} - \gamma\right)u\right\} du\right]$$

$$= O\left\{(1-\gamma w)^{-(\alpha+2)} \int_{(A/w)-\delta\gamma}^{\infty} v^{\alpha+1} e^{-v}\, dv\right\}$$

$$\to 0 \text{ as } A \to +\infty, \text{ uniformly in } x\ (0 \leqslant x \leqslant \delta/A). \quad (2.5)$$

Combining (2.1), (2.2), (2.3), (2.4), and (2.5) we see that, for each fixed $w > 0$,

$$\int_0^1 \left\{\frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_{Ax}^{\infty} s(u) u^{\alpha} e^{-u/wx}\, du\right\} d\phi(x) = o\left(\int_0^1 |d\phi(x)|\right) \to 0 \quad \text{as } A \to +\infty. \quad (2.6)$$

By supposition $s(x) = 0$ for $0 \leqslant x \leqslant 1$. Therefore

$$\int_0^1 \left\{ \frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^1 s(u) u^\alpha e^{-u/wx}\, du \right\} d\phi(x) = 0 \quad \text{for each } w > 0. \tag{2.7}$$

Now, by (2.6) and (2.7),

$$\begin{aligned}
&\int_0^1 \left\{ \frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_0^\infty s(u) u^\alpha e^{-u/wx}\, du \right\} d\phi(x) \\
&\qquad = \lim_{A \uparrow \infty} \int_0^1 \left\{ \frac{(wx)^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_1^{Ax} s(u) u^\alpha e^{-u/wx}\, du \right\} d\phi(x) \\
&\qquad = \lim_{A \uparrow \infty} \left\{ \int_0^1 \frac{w^{-(\alpha+1)}}{\Gamma(\alpha+1)} \int_1^A s(xv) v^\alpha e^{-v/w}\, dv \right\} d\phi(x) \\
&\qquad = \lim_{A \uparrow \infty} \int_1^A \left\{ \int_0^1 s(xv)\, d\phi(x) \right\} \frac{w^{-(\alpha+1)}}{\Gamma(\alpha+1)} v^\alpha e^{-v/w}\, dv \\
&\qquad = L^{(\alpha)}\left\{ \frac{1}{w}, t(v) \right\} \quad \text{for each } w > 0.
\end{aligned}$$

Here (2.8) is the identity given in Theorem (2.1). Now, we may look on the left-hand side of (2.8) as a regular $[H, \phi(x)]$ transform of $L^{(\alpha)}(1/w)$, which completes the proof since

$$\lim_{w \to \infty} L^{(\alpha)}(w^{-1}) = s.$$

3. An inclusion relation for (L, α) methods of summability

The main result of this section is

THEOREM 3.1. *Suppose that $L^{(\beta)}\{\tau, s(x)\}$ exists for all $\tau > 0$ and some fixed $\beta > -1$. Then $L^{(\alpha)}\{\tau, s(x)\}$ exists for all $\tau > 0$ and all α satisfying $\beta > \alpha > -1$. We also have, for these values of τ, α,*

$$\frac{\Gamma(\beta+1)\tau^{\alpha+1}}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)} \int_\tau^\infty (t-\tau)^{\beta-\alpha-1} t^{-(\beta+1)} L^{(\beta)}(t)\, dt = L^{(\alpha)}(\tau). \tag{3.1}$$

If, in addition, $s(x)$ is summable (L, β) to s, then $s(x)$ is summable (L, α) to s for each α satisfying $\beta > \alpha > -1$.

Theorem 3.1 for absolutely convergent integrals for $\beta > 0$ and $\alpha = 0$ is Theorem A′ of (9).

We need the easily proved lemma:

LEMMA 3.1. *For any four real numbers β, α, τ, u satisfying $\beta > \alpha$ and τ, $u > 0$*

$$\frac{\Gamma(\beta+1)\tau^{\alpha+1}}{\Gamma(\beta-\alpha)\Gamma(\alpha+1)} \int_{\tau}^{\infty} (t-\tau)^{\beta-\alpha-1} t^{-(\beta+1)}\, dt = 1$$

and

$$\int_{\tau}^{\infty} (t-\tau)^{\beta-\alpha-1} e^{-tu}\, dt = \Gamma(\beta-\alpha) u^{-(\beta-\alpha)} e^{-\tau u}.$$

Proof of Theorem 3.1. There exists a positive constant D such that $|s(x)| \leqslant D$ for $0 \leqslant x \leqslant 1$. Thus, for $\tau > 0$,

$$\left| \int_{0}^{1} s(t) t^{\beta} e^{-\tau t}\, dt \right| \leqslant D \int_{0}^{1} t^{\beta} e^{-\tau t}\, dt \leqslant D\Gamma(\beta+1)\tau^{-(\beta+1)}. \tag{3.2}$$

Now (3.2) implies that the repeated integral

$$\int_{\tau}^{\infty} (t-\tau)^{\beta-\alpha-1} \left(\int_{0}^{1} s(u) u^{\beta} e^{-tu}\, du \right) dt \tag{3.3}$$

exists for all $\tau > 0$. Integration by parts and the estimation of $s_1(x)$ at the beginning of the proof of Theorem 2.1 yields, for any fixed $a > 0$,

$$\begin{aligned} \int_{a}^{\infty -} s(t) t^{\beta} e^{-\tau t}\, dt &= [s_1(t) t^{\beta} e^{-\tau t}]_{t=a}^{t=\infty} - \int_{a}^{\infty -} s_1(t)\{\beta - \tau t\} t^{\beta-1} e^{-\tau t}\, dt \\ &= -s_1(a) e^{-\tau} a^{\beta} - \beta \int_{a}^{\infty -} s_1(t) t^{\beta-1} e^{-\tau t}\, dt + \tau \int_{a}^{\infty -} s_1(t) t^{\beta} e^{-\tau t}\, dt \\ &= O(e^{-\frac{1}{2}\sigma\tau}), \quad \text{as} \quad \tau \uparrow \infty. \end{aligned} \tag{3.4}$$

By (3.4) the repeated integral

$$\int_{\tau}^{\infty -} (t-\tau)^{\beta-\alpha-1} \left(\int_{1}^{\infty -} s(u) u^{\beta} e^{-\tau u}\, du \right) dt$$

exists for all $\tau > 0$. Therefore the repeated integral

$$\int_{\tau}^{\infty -} (t-\tau)^{\beta-\alpha-1} \left(\int_{0}^{\infty -} s(u) u^{\beta} e^{-\tau u}\, du \right) dt$$

exists for all $\tau > 0$. If it were allowable to change the order of integration in the last repeated integral, we should have, by Lemma 3.1,

$$\begin{aligned} \int_{\tau}^{\infty -} (t-\tau)^{\beta-\alpha-1} \left(\int_{0}^{\infty -} s(u) u^{\beta} e^{-\tau u}\, du \right) dt &= \int_{0}^{\infty -} s(u) u^{\beta} \left(\int_{\tau}^{\infty -} (t-\tau)^{\beta-\alpha-1} e^{-tu}\, dt \right) du \\ &= \Gamma(\beta-\alpha) \int_{0}^{\infty -} s(u) u^{\alpha} e^{-\tau u}\, du, \end{aligned}$$

for all $\tau > 0$. The last identity is the same as the first conclusion of the theorem. We have to show, then, that we may change the order of integration. Now, for $0 < X, Y < +\infty$,

$$J \equiv \int_\tau^{\infty-} (t-\tau)^{\beta-\alpha-1}\left\{\int_0^{\infty-} s(u)u^\beta e^{-tu}\,du\right\}dt$$

$$= \int_X^{\infty-} (t-\tau)^{\beta-\alpha-1}\left\{\int_0^{\infty-} s(u)u^\beta e^{-tu}\,du\right\}dt +$$

$$+ \int_\tau^X (t-\tau)^{\beta-\alpha-1}\left\{\int^Y s(u)u^\beta e^{-tu}\,du\right\}dt +$$

$$+ \int_\tau^X (t-\tau)^{\beta-\alpha-1}\left\{\int_Y^{\infty-} s(u)u^\beta e^{-tu}\,du\right\}dt$$

$$\equiv I_1+I_2+I_3, \quad \text{say.}$$

We have $\lim_{X\uparrow\infty} I_1 = 0.$

By (3.4) and Lemma 3.1,

$$I_3 \equiv \int_\tau^X (t-\tau)^{\beta-\alpha-1}\left\{\int_Y^{\infty-} s(u)u^\beta e^{-tu}\,du\right\}dt$$

$$= O\left\{\int_\tau^\infty (t-\tau)^{\beta-\alpha-1}e^{-\frac{1}{2}Yt}\,dt\right\} \quad (\text{as } Y\uparrow\infty)$$

$$= O\{Y^{-(\beta-\alpha)}e^{-\frac{1}{2}\tau Y}\} \quad (\text{as } Y\uparrow\infty).$$

Therefore $\lim_{Y\uparrow\infty} I_3 = 0,$

and
$$J = \lim_{X\uparrow\infty, Y\uparrow\infty} \int_\tau^X (t-\tau)^{\beta-\alpha-1}\left\{\int_0^Y s(u)u^\beta e^{-tu}\,du\right\}dt. \tag{3.5}$$

By Hobson's generalization of Fubini's theorem [(6) 631] we may invert the order of integration on the right of (3.4), so that

$$J = \lim_{X\uparrow\infty, Y\uparrow\infty} \int_0^Y s(u)u^\beta\left\{\int_\tau^X (t-\tau)^{\beta-\alpha-1}e^{-tu}\,dt\right\}du. \tag{3.6}$$

It is easy to see, by the second mean-value theorem for integrals, that as $X\uparrow\infty$

$$\int_\tau^X (t-\tau)^{\beta-\alpha-1}e^{-tu}\,dt \to \int_\tau^\infty (t-\tau)^{\beta-\alpha-1}e^{-tu}\,dt,$$

uniformly in u $(0 \leqslant \epsilon \leqslant u \leqslant Y < \infty)$. So that, since $s(u)$ is bounded in $0 \leqslant u \leqslant \epsilon$,

$$J = \lim_{Y \uparrow \infty} \int_0^Y s(u) u^\beta \left\{ \int_\tau^\infty (t-\tau)^{\beta-\alpha-1} e^{-tu}\, dt \right\} du$$

and, by Lemma 3.1,

$$= \Gamma(\beta-\alpha) \lim_{Y \uparrow \infty} \int_0^Y s(u) u^\alpha e^{-\tau u}\, du$$

$$= \Gamma(\beta-\alpha) \int_0^{\infty-} s(u) u^\alpha e^{-tu}\, du$$

for all $\tau > 0$. This proves the first assertion of the theorem. It is easy to see that the left-hand side of (3.1) is a regular transformation of $L^{(\beta)}(\tau)$ into $L^{(\alpha)}(\tau)$, a fact which completes the proof of the theorem.

4. The $M(\alpha, \beta)$ transformations

By $\int_{0+} f(x)\, dx$ we mean $\lim_{\epsilon \downarrow 0} \int_\epsilon f(x)\, dx$ whenever the last limit exists.

Define the function $\chi_\alpha(x)$, for $\alpha \geqslant 1$, by

$$\chi_\alpha(x) = \begin{cases} 0 & (0 \leqslant x < \alpha^{-1}), \\ 1 & (\alpha^{-1} \leqslant x \leqslant 1). \end{cases}$$

For any two real numbers α and β satisfying $\beta > \alpha \geqslant 1$ and for any function $s(x)$ bounded and L-integrable in every finite interval of $x \geqslant 0$ we call the function $M^{(\alpha,\beta)}(x)$ defined by

$$M^{(\alpha,\beta)}(x) = \int_0^1 s(xt) \frac{\chi_\alpha(t) - \chi_\beta(t)}{(\log \beta/\alpha)t}\, dt \quad (x \geqslant 0),$$

the '$M(\alpha, \beta)$ transform of $s(x)$'. The $M(\alpha, \beta)$ transformation is a Hausdorff transformation. It is easy to prove the theorems:

THEOREM 4.1. *For $\beta > \alpha \geqslant 1$ and $x \geqslant 0$,*

$$M^{(\alpha,\beta)}(x) = \int_{x/\beta}^{x/\alpha} \frac{s(u)}{(\log \beta/\alpha)u}\, du.$$

THEOREM 4.2. *For $\beta > \alpha \geqslant 1$ and $y > x > 0$,*

$$M^{(\alpha,\beta)}(y) - M^{(\alpha,\beta)}(x) = \int_x^y \frac{s(t/\alpha) - s(t/\beta)}{(\log \beta/\alpha)t}\, dt.$$

If, in addition, $s(0+)$ exists, then, for $x \geqslant 0$,

$$M^{(\alpha,\beta)}(x) = s(0+) + \int_{0+}^{x} \frac{s(t/\alpha) - s(t/\beta)}{(\log \beta/\alpha)t}\, dt.$$

The $M(\alpha, \beta)$ transforms appear in a proof by Hardy and Littlewood of R. Schmidt's tauberian theorem [see (4) 164–5].

Two interesting properties of Frullani integrals which we shall use later are given in the following propositions due to R. P. Agnew. [For the first see (2) Theorem 4, and for the second (1) Theorem 6.2.]

THEOREM A. *Let $f(t)$ be Lebesgue integrable over each finite interval of $x > 0$. If the Frullani integral on the left of*

$$\int_{0+}^{\infty-} \frac{f(at) - f(bt)}{t}\, dt = \lambda \left[\lim_{x \uparrow \infty} \frac{1}{x} \int_{0+}^{x} f(t)\, dt - \lim_{x \downarrow 0} \frac{1}{x} \int_{0+}^{x} f(t)\, dt \right] \tag{4.1}$$

exists for each $\lambda = \log(a/b)$ in a set of positive measure, then the mean values on the right exist and (4.1) holds for each pair of positive numbers a and b. On the other hand, if the mean values on the right of (4.1) exist, the left-hand member exists and the formula holds for each pair of positive numbers a and b.

THEOREM B. *Let $f(t)$ be Lebesgue integrable over each finite interval of $x > 0$. If*

$$\lim_{x \uparrow \infty} \int_{x}^{\lambda x} \frac{f(u)}{u}\, du$$

exists for some $\lambda > 0$, then

$$\lim_{x \uparrow \infty} \frac{1}{\log x} \int_{1}^{x} \frac{f(u)}{u}\, du$$

exists and

$$\lim_{x \uparrow \infty} \int_{x}^{\lambda x} \frac{f(u)}{(\log \lambda)u}\, du = \lim_{x \uparrow \infty} \frac{1}{\log x} \int_{1}^{x} \frac{f(u)}{u}\, du.$$

5. Some tauberian theorems

We need

THEOREM 5.1. *Suppose that*

(i) *$s(x)$ is a function bounded and L-integrable in every finite interval of $x \geqslant 0$;*

(ii) *for some $\lambda > 1$, $M > 0$, and a finite c,*

$$\liminf_{x \to \infty} \inf_{x < y \leqslant \lambda x} \{s(y) - Ms(x)\} \geqslant -c;$$

(iii) *$\phi(u) > 0$ is defined and L-integrable in every finite interval of $u \geqslant 0$;*

(iv) $\Phi(u) \equiv \int_0^u \phi(v)\,dv \to +\infty \quad as\ u \to +\infty;$

(v) $\lim_{\delta \downarrow 1} \lim_{x \to \infty} \frac{\Phi(\delta x)}{\Phi(x)} = 1;$

(vi) $\tau(x) \equiv \{\Phi(x)\}^{-1} \int_0^x s(u)\phi(u)\,du \to s \quad as\ x \to \infty.$

Then
$$Ms - c \leqslant \lim_{x \to \infty} \inf_{\sup} s(x) \leqslant \frac{s+c}{M}. \tag{5.1}$$

If, in addition, $(M-1)s = c$, *then*
$$\lim_{x \to \infty} s(x) = s.$$

Inequalities (5.1) *are best-possible.*

Proof. For all $y > x > 0$ we have
$$Ms(x) = \tau(y) + \frac{\Phi(x)}{\Phi(y)-\Phi(x)}[\tau(y)-\tau(x)] - \\ - \frac{1}{\Phi(y)-\Phi(x)} \int_x^y \{s(t)-Ms(x)\}\phi(t)\,dt, \tag{5.2}$$
$$M^{-1}s(y) = \tau(y) + \frac{\Phi(x)}{\Phi(y)-\Phi(x)}[\tau(y)-\tau(x)] + \\ + \frac{1}{M}\frac{1}{\Phi(y)-\Phi(x)} \int_x^y \{s(y)-Ms(t)\}\phi(t)\,dt. \tag{5.2'}$$

For some $\epsilon > 0$ choose $x_1(\epsilon) > 0$ and $\delta_1(\epsilon)$ $(1 < \delta_1(\epsilon) < \lambda)$, so that, for all $y > x > x_1$ and $1 < \delta < \delta_1$,
$$\left|\frac{\Phi(\delta x)}{\Phi(x)} - 1\right| \leqslant \sqrt{\epsilon}, \qquad |\tau(y)-\tau(x)| \leqslant \epsilon$$
and
$$\inf_{x < u \leqslant \lambda x} \{s(u)-Ms(x)\} \geqslant -(c+\epsilon).$$

Choose $y = \delta x$ in (5.2). Then, for $y > x > x_1(\epsilon)$ and $1 < \delta < \delta_1(\epsilon)$,
$$Ms(x) = \tau(\delta x) + \left\{\frac{\Phi(\delta x)}{\Phi(x)} - 1\right\}^{-1}\{\tau(\delta x)-\tau(x)\} - \\ - \{\Phi(\delta x)-\Phi(x)\}^{-1} \int_x^y \{s(t)-Ms(x)\}\phi(t)\,dt$$
$$\leqslant \tau(\delta x) + \sqrt{\epsilon} + c + \epsilon.$$

Hence
$$M \limsup_{x \to \infty} s(x) \leqslant s + c + \epsilon + \sqrt{\epsilon}.$$

The last inequality is valid for each $\epsilon > 0$; therefore
$$\limsup_{x \to \infty} s(x) \leqslant \frac{s+c}{M}.$$

Similarly we obtain from (5.2')

$$\liminf_{x\to\infty} s(x) \geqslant Ms-c.$$

If $(M-1)s = c$, then $Ms-c = M^{-1}(s+c)$, and we obtain $\lim_{x\to\infty} s(x) = s$. Choose a sequence $\{x_k\}$ $(k = 0, 1, 2, \ldots)$ in such a way that

$$\lim_{k\to\infty} \frac{x_{k+1}}{x_k} = +\infty \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{\Phi(x)} \sum_{x_k<x} \int_{x_k}^{\min(x, x_k+1)} \phi(t)\, dt = 0.$$

By (v) and (iv), this is possible. Now define $\tilde{s}(x)$ by

$$\tilde{s}(x) = \begin{cases} 1 \ (x_{2k} \leqslant x < x_{2k}+1), \\ -a \text{ for some } a > 0 \ (x_{2k+1} \leqslant x < 1+x_{2k+1}), \\ 0 \text{ (for all other values of } x). \end{cases}$$

We have then

$$\lim_{x\to\infty} \frac{1}{\Phi(x)} \int_0^x \tilde{s}(y)\phi(y)\, dy = 0,$$

$$\limsup_{x\to\infty} \tilde{s}(x) = 1, \qquad \liminf_{x\to\infty} \tilde{s}(x) = -a$$

and, for any fixed $\lambda > 1$ and $M > 0$,

$$\liminf_{x\to\infty} \inf_{x<y\leqslant\lambda x} \{\tilde{s}(y) - M\tilde{s}(x)\} \geqslant -\max(M, a) \equiv -c,$$

$$\frac{s+c}{M} = \frac{\max(M, a)}{M}, \qquad Ms-c = -\max(M, a).$$

If we choose $M > a$, then $(s+c)M^{-1} = \limsup_{x\to\infty} \tilde{s}(x)$. If we choose $M < a$, then $Ms-c = \liminf_{x\to\infty} \tilde{s}(x)$. In particular, for $M = a$,

$$Ms-c = \liminf_{x\to\infty} \tilde{s}(x), \qquad \frac{s+c}{M} = \limsup_{x\to\infty} \tilde{s}(x).$$

We may remark here that, if condition (1.2) is satisfied by a function $s(x)$, then for each $c' > c$ we can find a number $\lambda' = \lambda'(c') > 1$ such that (1.1) is satisfied for $\lambda = \lambda'$ and $c = c'$. We also see that the condition

$$Ms-c \leqslant \lim_{x\to\infty} \frac{\inf}{\sup} s(x) \leqslant \frac{s+c}{M}$$

is equivalent to the condition

$$Ms-c' \leqslant \lim_{x\to\infty} \frac{\inf}{\sup} s(x) \leqslant \frac{s+c'}{M}$$

for each $c' > c$. Therefore the conclusion of Theorem 5.1 remains valid if condition (ii) is replaced by (1.2). The same remark applies to all the other tauberian theorems in this section.

We may remark here too that (1.2) with $M = 1$ and $c = 0$ is R. Schmidt's tauberian condition and in this case we always have $(M-1)s = c$.

Definition. Suppose that the function $s(x)$ satisfies

$$s(x) = s(0) + \int_0^x f(t)\,dt \quad (x > 0), \tag{5.3}$$

where $f(x)$ is bounded and L-integrable in each finite interval of $x \geqslant 0$. For each number $\alpha > -1$ and for all $x \geqslant 0$, $\sigma^{(\alpha)}(x)$ is defined by

$$\sigma^{(\alpha)}(x) = s(0) + x^{-\alpha} \int_0^x (x-t)^{\alpha} f(t)\,dt.$$

It is easy to see that the new definition of $\sigma^{(\alpha)}(x)$ coincides with the previous one for $\alpha \geqslant 0$.

Convention. In order to avoid stating many propositions twice we make the convention that, if a Cesàro transform of an order between -1 and 0 appears in the statement of a theorem, then for this case we should add supposition (5.3) to the suppositions of the theorem.

If in Theorem 4.1 we choose $\phi(t) = t^{\alpha}$ for $\alpha > -1$ and replace $s(x)$ by $\sigma^{(\alpha)}(x)$, we obtain

THEOREM 5.2. *Suppose that, for some* $\alpha > -1$,

$$\lim_{x\to\infty} \sigma^{(\alpha+1)}(x) = s.$$

If

$$\liminf_{x\to\infty} \inf_{x<y\leqslant\lambda x} \{\sigma^{(\alpha)}(y) - M\sigma^{(\alpha)}(x)\} \geqslant -c \tag{5.4}$$

for some $\lambda > 1$, $M > 0$, *and finite* c *then*

$$Ms - c \leqslant \liminf_{x\to\infty} \sigma^{(\alpha)}(x) \leqslant \limsup_{x\to\infty} \sigma^{(\alpha)}(x) \leqslant \frac{s+c}{M}, \tag{5.5}$$

and $s(x)$ *is summable* $(C, \alpha+\epsilon)$ *for each* $\epsilon > 0$. *If, in addition,* $(M-1)s = c$ *then* $\lim_{x\to\infty} \sigma^{(\alpha)}(x) = s$. *Inequalities* (5.5) *are best-possible.*

The case $\alpha = 0$ of the last theorem is a generalization of Theorem C of (3) by R. C. Buck, and therefore [see (3)] of some tauberian theorems for quasimonotone and monotone sequences.

We need the lemma:

LEMMA 5.1. *Suppose that* $s(x)$ *satisfies* (ii) *of Theorem* 5.1. *Let* $\chi(x)$ *be a non-decreasing function in* $0 \leqslant x \leqslant 1$, *satisfying*

$$\chi(0) = \chi(0+) = 0, \qquad \chi(1) = 1.$$

If
$$\tau(x) \equiv \int_0^1 s(xt)\, d\chi(t) \quad (x \geqslant 0),$$

then
$$\liminf_{x\to\infty} \inf_{x<y\leqslant\lambda x} \{\tau(y) - M\tau(x)\} \geqslant -c.$$

The special case $\chi(x) = 1-(1-x)^\alpha$ $(\alpha > 0)$, $M = 1$, of Lemma 5.1 was given by C. T. Rajagopal [see (**10**) Lemma 1]. The proof of Lemma 5.1 is exactly the same as that of its special case.

From Theorem 4.2, Theorem 4.4, and Lemma 4.1 we obtain immediately the theorem:

THEOREM 5.3. *If $s(x)$ is summable Cesàro to s (that is, for some $\beta > -1$, $\lim_{x\to\infty} \sigma^{(\beta)}(x) = s$) and* (5.4) *is satisfied for some $\alpha > -1$, then* (5.5) *is satisfied, and $s(x)$ is summable $(C, \alpha+\epsilon)$ to s for each $\epsilon > 0$. If, in addition, $(M-1)s = c$ then $\lim_{x\to\infty} \sigma^{(\alpha)}(x) = s$.*

I now state one of the main results of this paper.

THEOREM 5.4. *Suppose $s(x)$ summable (L, α) to s for some $\alpha > -1$. If condition* (ii) *of Theorem* 5.1 *is satisfied, then* (5.1) *is satisfied and $s(x)$ is summable (C, ϵ) to s for each $\epsilon > 0$. If, in addition, $(M-1)s = c$, then $\lim_{x\to\infty} s(x) = s$.*

If for a sequence $\{s_n\}$ we define $s(x)$ by $s(x) = s_n$ for $n \leqslant x < n+1$, we obtain from Theorem 5.4 the theorem:

THEOREM 5.5. *If a sequence $\{s_n\}$ is Abel-summable to s: that is,*
$$\lim_{x\uparrow 1} (1-x) \sum_{n=0}^{\infty} s_n x^n = s$$
and, for some $\lambda > 1$, $M > 0$, and finite c,
$$\liminf_{n\to\infty} \min_{n<m\leqslant\lambda n} (s_m - Ms_n) \geqslant -c,$$
then
$$Ms - c \leqslant \lim_{n\to\infty} \inf \sup s_n \leqslant \frac{s+c}{M}, \tag{5.6}$$
and $\{s_n\}$ is summable (C, ϵ) to s for each $\epsilon > 0$. If, in addition, $(M-1)s = c$, then $\lim_{n\to\infty} s_n = s$. Inequalities (5.6) *are best-possible.*

We need the following theorem:

THEOREM H. *If, for some fixed $\alpha > -1$, $s(x)$ is summable (L, α) to s and $s(x) = O_L(1)$, as $x \to \infty$, then $\lim_{x\to\infty} \sigma^{(1)}(x) = s$.*

Theorem H is an immediate consequence of Lemma 2 of (**9**) and Theorem 1 of (**5**) of Hardy and Littlewood with $f(t)$ replaced by $t^\alpha s(t)$

and α replaced by $\alpha+1$ in its statement. Theorem 1 was first stated and attributed to Hardy and Littlewood by Titchmarsh in (**15**) and later stated by Hardy and Littlewood in (**5**); but they did not publish the details of its proof since this result was substantially superseded by R. Schmidt's then recently published results. For Theorem 1 see also Theorem HL of (**9**).

Proof of Theorem 5.4. For $M = 1$ it was proved by T. Vijayaraghavan [see (**16**)] that $s(x)$ is bounded. The rest of the proof of this case follows now from Theorem H, Theorem 5.2, and Theorem 5.4. If $0 < M < 1$, then for a fixed $x > 0$ and for all t satisfying

$$\lambda^{k-1}x < t \leqslant \lambda^k x \quad (k = 1, 2, \ldots),$$

$$s(t) - M^k s(x) \geqslant -(1+M+\ldots+M^{k-1})c.$$

Accordingly $$\liminf_{t\to\infty} s(t) \geqslant -(1-M)^{-1}c > -\infty.$$

Thus $s(x)$ is bounded on one side, and the rest of the proof of this case is the same as that for $M = 1$. If $M > 1$, I shall show that we cannot have $\limsup_{x\to\infty} s(x) = +\infty$. Suppose that there is a sequence $\{x_n\}$ ascending to infinity such that $\lim_{n\to\infty} s(x_n) = +\infty$. We can choose x_n so large that

$$Ms(x_n) - (1-M^{-1})^{-1}c > 0.$$

Then, for all values of u satisfying $\lambda^{k-1}x_n < u \leqslant \lambda^k x_n$ $(k = 1, 2, \ldots)$,

$$s(u) - M^k s(x_n) \geqslant -(M^k-1)(M-1)^{-1}c.$$

It follows that

$$s(u) \geqslant M^{k-1}\{Ms(x_n) - (1-M^{-k})(1-M^{-1})^{-1}\}c,$$

and so $$\liminf_{u\to\infty} s(u) = +\infty.$$

Therefore $s(x) \to +\infty$ (L, α), which is a contradiction. Thus

$$\limsup_{x\to\infty} s(x) < +\infty$$

and the rest of the proof is the same as that of the case $M = 1$.

THEOREM 5.6. *Suppose $s(x)$ summable (L, α) to s for some $\alpha > -1$. If for some $\beta > -1$, $M > 0$, $\lambda > 1$ and a finite c,*

$$\liminf_{x\to\infty} \inf_{x<y\leqslant\lambda x} \{\sigma^{(\beta)}(y) - M\sigma^{(\beta)}(x)\} \geqslant -c,$$

then $$Ms - c \leqslant \lim_{x\to\infty} \inf_{\sup} \sigma^{(\beta)}(x) \leqslant \frac{s+c}{M},$$

and $s(x)$ is summable $(C, \beta+\epsilon)$ to s for each $\epsilon > 0$. If, in addition, $(M-1)s = c$, then $s(x)$ is summable (C, β) to s.

6. An additional tauberian theorem

The main result of this section is the theorem:

THEOREM 6.1. *Suppose $s(x)$ summable (L, α) to s for some $\alpha > -1$. If $s(x)$ satisfies* (1.3), *then*

$$\lim_{x\to\infty} \frac{1}{\log x} \int_1^x \frac{s(u)}{u}\, du = s. \tag{6.1}$$

If, in addition, $s(x)$ satisfies (1.4) *then*

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x s(u)\, du = s. \tag{6.2}$$

We need the following proposition:

THEOREM 6.2. *Suppose $s(x)$ summable (L, α) to s for some $\alpha > -1$. If $s(x)$ satisfies* (5.3) *and $xf(x) = O_L(1)$ as $x \to \infty$, then $\lim_{x\to\infty} s(x) = s$. If, in addition, $xf(x) = O(1)$, as $x \to \infty$, then $s(x)$ is summable $(C, -1+\epsilon)$ to s for each $\epsilon > 0$.*

In the proof of Theorem 6.2 we use the lemma:

LEMMA 6.1. *Suppose that $|s(x)| \leqslant A < +\infty$ for $x \geqslant x_0 > 0$. Let $\phi(x)$ be L-integrable in $0 \leqslant x \leqslant 1$. If $t(x)$ is defined by*

$$t(x) \equiv \int_0^1 s(xt)\phi(t)\, dt, \quad \text{for } x \geqslant 0,$$

then

$$\lim_{\lambda\downarrow 1} \limsup_{x\to\infty} \sup_{x<y\leqslant\lambda x} |t(y)-t(x)| = 0.$$

Lemma 6.1 is a special case of a result proved in an equivalent form in (4) § 12.6, 293.

Proof of Theorem 6.2. If $tf(t) = O_L(1)$, as $t \to \infty$, then

$$\lim_{\lambda\downarrow 1} \liminf_{x\to\infty} \inf_{x<y\leqslant\lambda x} \{s(y)-s(x)\} \geqslant 0, \tag{6.3}$$

which, by Theorem 5.6, shows that $\lim_{x\to\infty} s(x) = s$. Now, for $\alpha > -1$,

$$s(x)-\sigma^{(\alpha)}(x) = \int_0^1 \{(xt)f(xt)\} \frac{1-(1-t)^\alpha}{t}\, dt. \tag{6.4}$$

From (6.4), Lemma 6.1, and the fact $tf(t) = O(1)$ as $t \to \infty$, we obtain

$$\lim_{\lambda\downarrow 1} \limsup_{x\to\infty} \sup_{x<y\leqslant\lambda x} |\{s(y)-\sigma^{(\alpha)}(y)\}-\{s(x)-\sigma^{(\alpha)}(x)\}| = 0. \tag{6.5}$$

From (6.5) and (6.3) we obtain, for each $\alpha > -1$,

$$\lim_{\lambda \downarrow 1} \limsup_{x \to \infty} \sup_{x < y \leqslant \lambda x} |\sigma^{(\alpha)}(y) - \sigma^{(\alpha)}(x)| = 0. \tag{6.6}$$

Then (6.6) and Theorem 5.6 yield the required second conclusion.

Proof of Theorem 6.1. We may suppose that $s(x) = 0$ for $0 \leqslant x \leqslant 1$. Now, by Theorem 2.1, the $M(1, \lambda)$ transform of $s(x)$ is summable (L, α) to s. By (1.3), Theorem 4.2, and Theorem 6.2, $s(x)$ is summable $M(1, \lambda)$ to s. Therefore (6.1) follows by Theorem 4.2 and Theorem B. Similarly we deduce (6.2) from (1.4) by the previous argument and Theorem A.

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A FUNCTION-THEORETIC SOLUTION OF CERTAIN INTEGRAL EQUATIONS (I)

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THE solution of a great variety of boundary problems for elliptic partial differential equations in two variables can be reduced to the solution of one-dimensional integral equations with kernels of logarithmic or Cauchy type. Many of these equations can be solved with the aid of the Fourier-transform theorem in the complex domain by using the method of Wiener and Hopf. There exists, however, an entirely different approach to such integral equations which seems to have been little exploited and will serve as a subject of this paper. The kernels mentioned are either fundamental solutions of the associated differential equations or derivatives of such solutions. As such, they are analytic functions of their arguments with isolated singularities. It is suggested, therefore, that, if the boundary data are analytic, one might transform the associated integral equations into functional equations between analytic functions. Certain two-dimensional potential problems were solved in this manner by Carleman (**1**) and (**2**) and our method is an extension of his. In this paper we study one type of integral equation to which the method is applicable, and it is intended that other types shall be discussed in the future.

We shall study here integral equations of the form

$$\int_0^\infty K(x-\xi)f(\xi)\,d\xi = g(x) \quad (x > 0), \tag{1}$$

where $g(x)$ is given and $f(x)$ is sought. The kernel K is of the form

$$K(w) = P(w)\log w + Q(w), \tag{2}$$

where $P(w)$ and $Q(w)$ are entire functions of w^2, and $\log w$ denotes that branch of the logarithm which is real when $\arg w = 0$. Let us suppose that the function $g(x)$ in (1) can be continued to an analytic function $g(z)$ by replacing x by $z = x+iy$. In view of the analytic character of $K(w)$, one would expect that $f(x)$ could be similarly continued to an analytic function $f(z)$, and we shall discuss this possibility.

Consider the function $F(z)$, defined for $-2\pi \leqslant \arg z \leqslant 0$ by the integral

$$F(z) = \int_0^\infty K(\xi-z)f(\xi)\,d\xi \qquad (\arg \xi = 0), \tag{3}$$

Quart. J. Math. Oxford (2), 9 (1958), 132-43.

where $f(\xi)$ is a solution of (1). $F(z)$ is clearly analytic in the open sector $-2\pi < \arg z < 0$ and is continuous in the closed sector

$$-2\pi \leqslant \arg z \leqslant 0.$$

Moreover, $F(z)$ can be continued to an analytic function on a Riemann surface with a logarithmic branch-point at the origin as we shall show. To this end, let z^- denote the image point of z on such a surface: that is, $z^- = z\exp(-2\pi i)$. Then (1) can be converted into a functional relation which connects $F(z)$ and $F(z^-)$, and this provides the continuation mentioned above. For $\arg z = 0$, $F(z)$ is the limit of the integral in (3) as $y \to 0^-$, while $F(z^-)$ is the limit of the same integral as $y \to 0^+$. In view of the presence of the logarithmic term in $K(w)$, these limits are not equal. In fact, for $\arg z = 0$, we have

$$F(z^-)+F(z) = 2\int_0^\infty K[|\xi-z|]f(\xi)\,d\xi = 2g(z), \tag{4}$$

$$F(z^-)-F(z) = -2\pi i\int_0^z P(\xi-z)f(\xi)\,d\xi. \tag{5}$$

Since $g(z)$ is analytic, (4) is the relation we desire and permits $F(z)$ to be continued to all sheets of the Riemann surface. This equation now holds for all $z \neq 0$ on this surface. $F(z)$, by construction, is analytic in $-2\pi < \arg z < 0$ whatever the choice of $g(z)$. Suppose now that $g(z)$ has isolated singularities at $z_1,\ldots$. Then from (4) it follows that $F(z)$ has singularities at all of the image points of $z_1,\ldots$ (but not at $z_1,\ldots$), and these together with $z = 0$ and $z = \infty$, are the only possible singularities of $F(z)$.

Equation (4) plus a knowledge of $F(z)$ as $|z| \to \infty$ is sufficient to determine $F(z)$ explicitly. The estimate of $F(z)$ for large z makes use of the behaviour of the kernel $K(w)$ near $w = \infty$. Observe that, for any real positive ξ, $|\xi-z| \to \infty$ with $-\pi < \arg(\xi-z) < +\pi$ as $|z| \to \infty$ in $-2\pi < \arg z < 0$. Thus, if $K(w) \sim A(w)$ as $|w| \to \infty$ $(-\pi < \arg w < +\pi)$, we have

$$F(z) \sim \int_0^\infty A(\xi-z)f(\xi)\,d\xi \quad \text{as } |z| \to \infty \ (-2\pi < \arg z < 0).$$

This relation should lead one to an asymptotic estimate for $F(z)$, which then needs verification after a solution is obtained. It will be observed that the estimate for $F(z)$ is the only non-routine step in our procedure. Suppose that we succeed in finding an entire function $\psi(z)$ such that

$\psi(z)F(z)$ is bounded in the closed sector $-2\pi \leqslant \arg z \leqslant 0$. Then we rewrite (4) as

$$\{F(z^-)\psi(z^-)/\sqrt{z^-}\}-\{F(z)\psi(z)/\sqrt{z}\} = -2g(z)\psi(z)/\sqrt{z}. \tag{6}$$

We shall now derive one solution of this equation. Consider the function $\phi(z)$ defined for $-2\pi \leqslant \arg z \leqslant 0$ by the integral

$$\phi(z) = \frac{1}{\pi i}\int_0^\infty \frac{g(t)\psi(t)}{(z-t)\sqrt{t}}\,dt \quad (\arg t = 0).$$

The function $\phi(z)$ is continuous in $-2\pi \leqslant \arg z \leqslant 0$ and analytic in $-2\pi < \arg z < 0$. Furthermore, when $\arg z = 0$ and -2π, $\phi(z)$ has the respective limit values $\phi(z)$ and $\phi(z^-)$ satisfying

$$\phi(z^-)-\phi(z) = -2g(z)\psi(z)/\sqrt{z}. \tag{7}$$

Equation (7) again provides the continuation of $\phi(z)$ to a logarithmic Riemann surface on which the equation holds for all z other than zero. If $g(z)$ has isolated singularities at $z_1,\ldots$, it is again true that $\phi(z)\sqrt{z}/\psi(z)$ has singularities only at image points of $z_1,\ldots$, at $z = 0$ and at $z = \infty$. Observe, moreover, that the singularities at image points of $z_1,\ldots$, are precisely the same as the corresponding ones for $F(z)$. We have thus shown that the difference

$$F(z)\psi(z)/\sqrt{z}-\phi(z) \tag{8}$$

is a single-valued analytic function in $0 < |z| < \infty$ whatever the singularities of $g(z)$. If $g(t)\psi(t)/\sqrt{t}$ is integrable in $[0,\infty]$, then $\phi(z) \to 0$ as $|z| \to \infty$ in $-2\pi \leqslant \arg z \leqslant 0$.† Our original choice of $\psi(z)$ is such that the same is true of $F(z)\psi(z)/\sqrt{z}$ hence the difference (8) vanishes as $|z| \to \infty$. If $g(t)$ is continuous at $t = 0$, then, from (2),

$$\phi(z) = O(1/\sqrt{z}) \quad \text{as } z \to 0 \ (-2\pi \leqslant \arg z \leqslant 0).$$

Further, $F(z)$ is continuous at $z = 0$, and hence

$$F(z)\psi(z)/\sqrt{z} = O(1/\sqrt{z}) \quad \text{as } z \to 0 \ (-2\pi \leqslant \arg z \leqslant 0).$$

The difference (8) is thus $O(1/\sqrt{z})$ at $z = 0$ and vanishes as $|z| \to \infty$. Hence, being single-valued, it is identically zero: that is,

$$F(z) = \phi(z)\sqrt{z}/\psi(z). \tag{9}$$

If we use (7) and (9) to determine the left-hand member of (5), we arrive at the Volterra integral equation,

$$\pi i\int_0^z P(\xi-z)f(\xi)\,d\xi = \phi(z)\sqrt{z}/\psi(z)-g(z). \tag{10}$$

† We shall subsequently verify this fact in a special case which will illustrate the general method of verification.

A formal solution of (10) is easily obtained with the aid of the Laplace transform.

In order to illustrate the details of the method we shall derive a solution of the integral equation governing the diffraction of a plane sound wave by a semi-infinite rigid screen [the Sommerfeld half-plane problem (3)]. The appropriate integral equation is

$$\int_0^\infty H_0^{(1)}[k|x-\xi|]f(\xi)\,d\xi = \exp(i\alpha x) \quad (x>0), \tag{11}$$

where k and α are real numbers and $-k<\alpha<k$; (11) is of the form of (1) since

$$H_0^{(1)}(w) = 2i\pi^{-1}J_0(w)\log w + Q(w),$$

where $Q(w)$ is a power series in w^2 and $J_0(w)$ is the regular Bessel function of order zero.

We first repeat the formal arguments above. For large $|w|$ and $-\pi \leqslant \arg w \leqslant +\pi$, $H_0^{(1)}(w)$ possesses the asymptotic development

$$H_0^{(1)}(w) \sim \{\exp(iw)/\sqrt{w}\}\{A+O(w^{-1})\} \quad (A \text{ a constant}).$$

If this estimate be substituted into the integral (3) defining $F(z)$, we find, formally,

$$F(z) \sim B\exp(-ikz), \qquad |z|\to\infty \quad (B \text{ a constant})$$

for $-2\pi < \arg z < 0$. The suggested choice of function $\psi(z)$ is thus $\exp(ikz)$, and we are led to

$$F(z) = \sqrt{z}\,\phi(z)\exp(-ikz),$$

where

$$\phi(z) = \int_0^\infty \frac{\exp[i(\alpha+k)t]}{\pi i(z-t)\sqrt{t}}\,dt.$$

The Volterra integral equation (5) thus becomes

$$\int_0^z J_0\{k(\xi-z)\}f(\xi)\,d\xi = -\tfrac{1}{4}\sqrt{z}[\phi(z^-)+\phi(z)]\exp(-ikz)$$

$$= \frac{\sqrt{z}\exp(-ikz)}{2\pi i}\int_0^\infty \frac{\exp[i(k+\alpha)t]}{(t-z)\sqrt{t}}\,dt \quad (z>0), \tag{12}$$

where the integral is intended as the Cauchy principal value.

In order to solve (12) we make a preliminary computation. Set $\beta = (k+\alpha)$ and consider the function

$$G(\beta,z) = \int_0^\infty \frac{\exp(i\beta t)}{(t-z)\sqrt{t}}\,dt = \int_0^\infty \frac{2\exp(i\beta u^2)}{u^2-z}\,du.$$

Differentiating† with respect to β we find

$$G'(\beta,z) = 2i \int_0^\infty \frac{u^2 \exp(i\beta u^2)}{u^2-z}\,du = 2i \int_0^\infty \exp(i\beta u^2)\,du + izG$$
$$= -i(1+i)\sqrt{\frac{\pi}{2\beta}} + izG.$$

When $\beta = 0$, it is readily verified that $G(\beta,z) = 0$. Hence

$$G(\beta,z) = i(1+i)\sqrt{\frac{\pi}{2\beta}}\{\exp(iz\beta)\}\int_0^\beta \frac{\exp(-iz\sigma)}{\sqrt{\sigma}}\,d\sigma.$$

We now apply the Laplace transform to (12). Writing $L(s)$ for the transform of $f(z)$, we have

$$\frac{L(s)}{\sqrt{(k^2+s^2)}} = \frac{1}{2\pi i}\int_0^\infty \sqrt{z}\exp(-sz-ikz)\,G(\beta,z)\,dz$$
$$= \frac{1+i}{2\pi}\sqrt{(\tfrac{1}{2}\pi)}\int_0^\beta \frac{d\sigma}{\sqrt{\sigma}}\int_0^\infty \sqrt{z}\exp\{-sz+i(\alpha-\sigma)z\}\,dz, \quad (13)$$

assuming the interchanges in integration are justified. Now, under the substitution $z = u^2$, the infinite integral in (13) becomes

$$2\int_0^\infty u^2\exp[u^2(-s+i\alpha-i\sigma)]\,du.$$

Hence, using the identity

$$\int_0^\infty u^2\exp(-au^2)\,du = -\frac{\partial}{\partial a}\int_0^\infty \exp(-au^2)\,du$$
$$= -\frac{\partial}{\partial a}a^{-\frac{1}{2}}\int_0^\infty \exp(-\tau^2)\,d\tau = \frac{\sqrt{\pi}}{4a^{\frac{3}{2}}},$$

equation (13) becomes

$$\frac{L(s)}{\sqrt{(k^2+s^2)}} = \frac{(1+i)}{4\sqrt{2}}\int_0^\beta \frac{d\sigma}{\sigma^{\frac{1}{2}}[s-i\alpha+i\sigma]^{\frac{3}{2}}} = \frac{i\sqrt{(\alpha+k)}}{2(\alpha+is)\sqrt{(k-is)}},$$

i.e.
$$L(s) = \frac{i\sqrt{\{(\alpha+k)(k+is)\}}}{2(\alpha+is)}.$$

† The formal step of differentiation is easily justified by first shifting the path of integration. See, for example, (4) 86.

The solution of (13) is then given by the inverse Laplace transform of $L(s)$,

$$f(z) = \frac{\sqrt{(\alpha+k)}}{4\pi} \int_{c-i\infty}^{c+i\infty} \frac{\sqrt{(k+is)}}{\alpha+is} \exp(sz)\, ds \tag{14}$$

for any positive c.

The function $f(z)$ defined by (14) agrees with the known solution (4) obtained by Wiener–Hopf methods. We shall, however, carry out the verification that $f(z)$ actually solves the original integral equation (11). This verification proceeds as follows. We show first that $f(z)$ is a solution of (11). Next we form the integral

$$F(z) = \int_0^\infty H_0^{(1)}\{k(\xi-z)\} f(\xi)\, d\xi \quad (-2\pi \leqslant \arg z \leqslant 0)$$

and observe that relation (5), that is

$$F(z^-) - F(z) = -4 \int_0^z J_0\{k(\xi-z)\} f(\xi)\, d\xi \quad \text{on } \arg z = 0 \tag{15}$$

still holds, assuming $f(\xi)$ sufficiently well-behaved at ∞. Then a comparison of (15) and (12) shows that

$$F(z^-) - F(z) = -\sqrt{z} \exp(-ikz)\{\phi(z^-) + \phi(z)\} \quad (\arg z = 0). \tag{16}$$

Since $\phi(z)$ is analytic on the logarithmic Riemann surface, (16) yields the analytic continuation of $F(z)$, and the equation continues to hold for all $z \neq 0$. Equation (16), however, implies that the difference,

$$\frac{F(z)}{\sqrt{z}} \exp(ikz) - \phi(z),$$

is single-valued and analytic in the entire z-plane with the possible exception of $z = 0$ and $z = \infty$. We shall show that this difference is $O(z^{-\frac{1}{2}})$ near $z = 0$ and tends to 0 as $z \to 0$, from which follows, as in the formal steps,

$$F(z) = \sqrt{z} \exp(-ikz)\, \phi(z). \tag{17}$$

Observe next that the first equality of (4), that is

$$F(z^-) + F(z) = 2 \int_0^\infty H_0^{(1)}(k|\xi-z|) f(\xi)\, d\xi \quad (\arg z = 0)$$

still holds. Hence, by (17),

$$2 \int_0^\infty H_0^{(1)}(k|\xi-z|) f(\xi)\, d\xi = -\sqrt{z} \exp(-ikz)\{\phi(z^-) - \phi(z)\}$$

and, by (7), $\quad \phi(z^-) - \phi(z) = -2 \exp\{i(\alpha+k)z\}/\sqrt{z}.$

Hence the verification will be complete. We proceed now to the details.

LEMMA 1. *$f(z)$ as defined by* (14) *satisfies* (12).

Proof. The transforms $L(s)$ and $1/\sqrt{(k^2+s^2)}$ of $f(z)$ and $J_0(kz)$ respectively exist in $\operatorname{re} s > 0$, and their product is the transform of the convolution

$$\int_0^z J_0\{k(\xi-z)\}f(\xi)\,d\xi$$

in this same region. By construction $L(s)/\sqrt{(k^2+s^2)}$ is the transform of the right-hand member of (12), which also exists in $\operatorname{re} s > 0$. Hence left-hand and right-hand members of (12) are equal.

LEMMA 2. *$f(z)$ as defined by* (14) *can be continued to an analytic function of z on a logarithmic Riemann surface.*

Proof. We observe first that $f(z)$, being a solution of (12), can be put in another form. Let $h(z)$ denote the right-hand member of (12) and $H(s)$ its transform. Then

$$L(s) = \sqrt{(k^2+s^2)}H(s) = (k^2+s^2)H(s)/\sqrt{(k^2+s^2)}.$$

As already remarked, if $g(t)$ is continuous at $t = 0, \phi(z) = O(z^{-\frac{1}{2}})$ as $z \to 0$. Hence, the function $h(z)$ is bounded for $z \to 0$, and accordingly $H(s) = O(s^{-1})$ ($|s| \to \infty$, $\operatorname{re} s > 0$). We rewrite the transform equation in the form

$$\frac{L(s)}{s} = \frac{\sqrt{(k^2+s^2)}}{s}H(s) = (k^2+s^2)\frac{H(s)}{s\sqrt{(k^2+s^2)}}.$$

The inverse Laplace transform of $H(s)/s$ is

$$\int_0^z h(\xi)\,d\xi = A(z),$$

and hence the last equation may be written

$$\int_0^z f(\xi)\,d\xi = \left(\frac{d^2}{dz^2}+k^2\right)\int_0^z J_0\{k(z-\xi)\}\,d\xi \int_0^\xi h(\xi')\,d\xi' \tag{18}$$

since $A(z)$ and its first derivative vanish as $z \to 0$. This relation holds initially on $\arg z = 0$. However, we have

$$h(z) = -\tfrac{1}{4}\sqrt{z}\exp(-ikz)\{\phi(z^-)+\phi(z)\}, \tag{19}$$

and, from the remarks made in conjunction with (7), it follows that $h(z)$, when defined by the right-hand side of (19), is analytic on the logarithmic Riemann surface [the function $g(z) = \exp(i\alpha z)$ has no singularities except

at ∞]. The right-hand member of (18) then provides a continuation of $f(z)$ as stated.

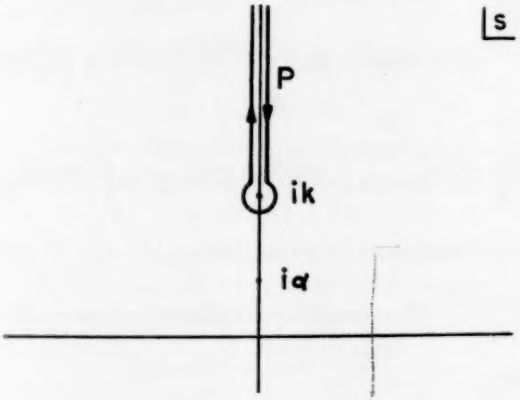
Remark. It should be observed that the argument given in Lemma 2 could be applied to an arbitrary analytic function $g(z)$. If $g(z)$ has isolated singularities at $z_1,\ldots$, then, as we have seen, $h(z)$, which is still given by (19), has singularities at $z = 0$, $z = \infty$, $z = z_1,\ldots$, and at all image points of $z_1,\ldots$. The same would be true of $f(z)$ by (18). For example, in the diffraction of a half-plane by a line source, the function $g(x)$ will possess at least a logarithmic singularity.

LEMMA 3. *For small positive ϵ, the function $f(z)$ defined by* (14) *satisfies*

$$f(z)-c\exp(i\alpha z) = O\{\exp(ikz)/z\}$$

as $|z| \to \infty$ in $0 \leqslant \arg z \leqslant \epsilon$, for some constant c.

Proof. For the sector stated the path of integration for $f(z)$ can be deformed into the path P indicated in the figure if one takes account of the pole at $s = i\alpha$.



Thus
$$f(z) = c_1\exp(i\alpha z)+\frac{i\sqrt{(\alpha+k)}}{4\pi}\int_P \frac{\sqrt{(s-ik)}}{\alpha+is}\exp(sz)\,ds$$

$$= c_1\exp(i\alpha z)+c_2\int_h^\infty \frac{\sqrt{(\eta-k)}}{\eta-\alpha}\exp(i\eta z)\,d\eta$$

$$= c_1\exp(i\alpha z)+c_2\exp(ikz)\int_0^\infty \frac{\sqrt{u}\exp(iuz)}{u+(k-\alpha)}\,du.$$

We have

$$\int_0^\infty \frac{\sqrt{u}\exp(iuz)}{u+(k-\alpha)}du = 2\int_0^\infty \frac{u^2\exp(iu^2z)}{u^2+(k-\alpha)}du$$

$$= 2\int_0^\infty \exp(iu^2z)\,du - 2(k-\alpha)\int_0^\infty \frac{\exp(iu^2z)}{u^2+(k-\alpha)}du.$$

Furthermore,

$$\frac{d}{dz}\int_0^\infty \frac{\exp(iu^2z)}{u^2+(k-\alpha)}du = 2i\int_0^\infty \exp(iu^2z)\,du - 2i(k-\alpha)\int_0^\infty \frac{\exp(iu^2z)}{u^2+(k-\alpha)}du.$$

Now,
$$2\int_0^\infty \exp(iu^2z)\,du = \frac{c_3}{\sqrt{z}},$$

and hence

$$\int_0^\infty \frac{\exp(iu^2z)}{u^2+(k-\alpha)}du$$

$$= ic_3\exp\{i(\alpha-k)z\}\int_0^z \frac{\exp\{i(k-\alpha)\xi\}}{\sqrt{\xi}}d\xi + c_4\exp\{i(\alpha-k))z\}.$$

Next we have

$$\int_0^z \frac{\exp\{i(k-\alpha)\xi\}}{\sqrt{\xi}}d\xi = \int_0^\infty \frac{\exp\{i(k-\alpha)\xi\}}{\sqrt{\xi}}d\xi - \int_z^\infty \frac{\exp\{i(k-\alpha)\xi\}}{\sqrt{\xi}}d\xi,$$

and successive integration by parts yields

$$\int_z^\infty \frac{\exp\{i(k-\alpha)\xi\}}{\sqrt{\xi}}d\xi = \exp\{i(k-\alpha)z\}(A_0z^{-\frac{1}{2}} + A_1z^{-\frac{3}{2}} + \ldots) \quad (k \neq \alpha).$$

Thus, we find

$$f(z) \sim c_1\exp(i\alpha z) + c_2\exp(ikz)[c_3z^{-\frac{1}{2}} - 2(k-\alpha)c_4\exp\{i(\alpha-k)z\} + \\ + A_0z^{-\frac{1}{2}} + A_1z^{-\frac{3}{2}} + \ldots],$$

which is the conclusion of the lemma.

LEMMA 4. *The function*

$$\phi(z) = \int_0^\infty \frac{\exp\{i(k+\alpha)t\}}{\pi i(z-t)\sqrt{t}}dt$$

satisfies $\phi(z) = O(z^{-\frac{1}{2}})$ *as* $|z| \to \infty$ *in* $-2\pi \leqslant \arg z \leqslant +\epsilon$ *for small positive* ϵ.

Proof. Consider first the sector $-2\pi+\epsilon \leqslant \arg z \leqslant -\epsilon$ for small positive ϵ. We have then

$$|z-t| \geqslant |t| \tan\epsilon, \qquad |z-t| \geqslant |z| \sin\epsilon$$

for $\arg t = 0$. We write

$$\int_0^\infty \frac{\exp\{i(k+\alpha)t\}}{(z-t)\sqrt{t}}\,dt = \int_0^H + \int_H^\infty$$

for any positive H. Clearly $\int_0^H = O(z^{-1})$ as $|z| \to \infty$. On the other hand,

$$\left|\int_H^\infty \frac{\exp\{i(k+\alpha)t\}}{(z-t)\sqrt{t}}\,dt\right| \leqslant \left|\frac{1}{(k+\alpha)(z-H)\sqrt{H}}\right| + \frac{1}{(k+\alpha)}\int_H^\infty \left|\frac{d}{dt}\left(\frac{1}{(z-t)\sqrt{t}}\right)\right| dt$$

on integration by parts. But

$$\begin{aligned}\left|\frac{d}{dt}\frac{1}{(z-t)\sqrt{t}}\right| &\leqslant \frac{1}{2t^{\frac{1}{2}}|z-t|} + \frac{1}{|t|^{\frac{1}{2}}|z-t|^2} \\ &\leqslant \frac{1}{|z|\,|t|^{\frac{1}{2}}\sin\epsilon} + \frac{1}{|z|\,|t|^{\frac{1}{2}}\sin\epsilon\tan\epsilon} \leqslant \frac{K}{|z|\,|t|^{\frac{1}{2}}},\end{aligned}$$

for some constant K. Hence $\int_H^\infty = O(z^{-1})$. Thus $\phi(z) = O(z^{-1})$ in

$$-2\pi+\epsilon \leqslant \arg z \leqslant -\epsilon.$$

Next observe that the path of integration defining $\phi(z)$ can be shifted into the upper half-plane so as to be along $\arg t \neq 0$ (> 0) since

$$\operatorname{re} i(k+\alpha)t < 0$$

in this region. The same argument as above can again be applied and we find $\phi(z) = O(z^{-1})$ in $-2\pi-\epsilon \leqslant \arg z \leqslant +\epsilon$ for small ϵ. To complete the estimate for $-2\pi \leqslant \arg z$ we use

$$\phi(z^-)-\phi(z) = \frac{\exp\{i(k+\alpha)z\}}{\sqrt{z}}.$$

We obtain $\phi(z)$ in $-2\pi \leqslant \arg z \leqslant -2\pi+\epsilon$ by computing $\phi(z)$ for z in $0 \leqslant \arg z \leqslant +\epsilon$. Thus

$$\phi(z) \leqslant z^{-\frac{1}{2}} + O(z^{-1}) \quad \text{in } -2\pi \leqslant \arg z \leqslant -\pi+\epsilon,$$

and the lemma is proved.

The last remaining step is the verification that, if $F(z)$ is defined by

$$F(z) = \int_0^\infty H_0^{(1)}\{k(\xi-z)\} f(\xi)\,d\xi \quad (-2\pi \leqslant \arg z \leqslant 0),$$

$$F(z^-)-F(z) = -\sqrt{z}\exp(ikz)\{\phi(z^-)+\phi(z)\}, \tag{20}$$

then

$$\exp(ikz)F(z) \text{ is bounded as } |z| \to \infty \text{ in } -2\pi \leqslant \arg z \leqslant 0.$$

The proof closely parallels that of Lemma 4. Consider again the sector $-2\pi+\epsilon \leqslant \arg z \leqslant -\epsilon$ for $\epsilon > 0$. As we have observed before,

$$|\xi - z| \to \infty \quad \text{with } -\pi < \arg(\xi - z) < +\pi$$

as $|z| \to \infty$ in this sector. The corresponding asymptotic estimate for $H_0^{(1)}$ yields

$$H_0^{(1)}\{k(\xi-z)\} - \frac{A}{\sqrt{(\xi-z)}}\exp\{ik(\xi-z)\} = O\left[\frac{\exp\{ik(\xi-z)\}}{|\xi-z|^{\frac{3}{2}}}\right].$$

Thus

$$F(z) - A\exp(-ikz)\int_0^\infty \frac{\exp(ik\xi)f(\xi)}{\sqrt{(\xi-z)}}\,d\xi$$

$$= \exp(ikz)\int_0^\infty K(\xi-z)\exp(ik\xi)f(\xi)\,d\xi,$$

where

$$|K(\xi-z)| \leqslant \frac{M}{|\xi-z|^{\frac{3}{2}}}$$

for some constant M. By Lemma 3 we have $|f(\xi)|$ bounded for $\arg \xi = 0$; hence,

$$\left|\int_0^\infty K(\xi-z)\exp(ik\xi)f(\xi)\,d\xi\right| \leqslant M'\int_0^1 \frac{d\xi}{|\xi-z|^{\frac{3}{2}}} + M'\int_1^\infty \frac{d\xi}{|\xi-z|^{\frac{3}{2}}}$$

$$\leqslant \frac{M''}{|z|^{\frac{3}{2}}} + \frac{M'}{|z|^{\frac{1}{2}}\sin^{\frac{1}{2}}\epsilon\tan^{\frac{1}{2}}\epsilon}\int_1^\infty \frac{d\xi}{|\xi|^{\frac{3}{2}}} = O(z^{-\frac{1}{2}}).$$

Next we have, by Lemma 3,

$$\int_0^\infty \frac{\exp(ik\xi)f(\xi)}{\sqrt{(\xi-z)}}\,d\xi = C\int_0^\infty \frac{\exp\{i(k+\alpha)\xi\}}{\sqrt{(\xi-z)}}\,d\xi + \int_0^\infty \frac{n(\xi)\exp(ik\xi)}{\sqrt{\{\xi(\xi-z)\}}}\,d\xi,$$

where $n(\xi)$ is bounded as $|\xi| \to \infty$. But integration by parts yields

$$\left|\int_0^\infty \frac{\exp\{i(k+\alpha)\xi\}}{\sqrt{(\xi-z)}}\,d\xi\right| \leqslant \frac{1}{(k+\alpha)\sqrt{z}} + \frac{1}{2(k+\alpha)}\int_0^\infty \frac{d\xi}{|\xi-z|^{\frac{3}{2}}} = O(z^{-\frac{1}{2}})$$

as above, and

$$\left|\int_0^\infty \frac{n(\xi)\exp(ik\xi)}{\sqrt{\{\xi(\xi-z)\}}}\,d\xi\right| \leqslant \left|\frac{n(0)}{(k+\alpha)\sqrt{z}}\right| + \frac{1}{(k+\alpha)}\int_0^\infty \frac{d}{d\xi}\left|\frac{1}{\sqrt{\xi}}\,\frac{n}{(\xi-z)}\right|d\xi.$$

Now

$$\frac{d}{d\xi}\left(\frac{n}{(\xi-z)\sqrt{\xi}}\right) = \frac{1}{(\xi-z)\sqrt{\xi}}\frac{dn}{d\xi} + n\frac{d}{d\xi}\left(\frac{1}{(\xi-z)\sqrt{\xi}}\right).$$

Since $n(\xi)$ is bounded as $|\xi| \to \infty$ and is analytic when

$$n = \{f - c\exp(i\alpha z)\}\sqrt{z}\exp(-ikz),$$

we have $dn(\xi)/d\xi = O(\xi^{-1})$, while

$$\left|\frac{d}{d\xi}\left(\frac{1}{(\xi-z)\sqrt{\xi}}\right)\right| \leqslant \frac{1}{2|\xi|^{\frac{1}{2}}|\xi-z|} + \frac{1}{(\xi-z)^2\sqrt{\xi}} \leqslant \frac{M}{\xi^{\frac{1}{2}}|z|},$$

as before. Thus $$\int_0^\infty \frac{n(\xi)\exp(ik\xi)}{\sqrt{\{\xi(\xi-z)\}}}\,d\xi = O(z^{-1}),$$

and $F(z) = O(z^{-\frac{1}{2}})$ and is certainly bounded as $|z| \to \infty$.

From Lemma 3 we see again that the path of integration defining $F(z)$ can be swung into the upper half-plane and the argument can be repeated to show that $F(z)$ is bounded in $-2\pi+\epsilon \leqslant \arg z \leqslant +\epsilon$. Finally, we use (20) to complete the estimate for $-2\pi \leqslant \arg z$. For

$$-2\pi \leqslant \arg z \leqslant -2\pi+\epsilon$$

we find $F(z)$ by computing $F(z^-)$ for $0 \leqslant \arg z \leqslant +\epsilon$. By Lemma 4 the right-hand member of (20) is bounded for $0 \leqslant \arg z \leqslant +\epsilon$ and as above $F(z)$ is bounded there. Consequently $F(z)$ is bounded in

$$-2\pi \leqslant \arg z \leqslant -2\pi+\epsilon.$$

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METRIZATION OF A SET WHICH IS MAPPED INTO ITSELF

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1. Summary

D. Ellis (1) raised the problem: *given a set S and a mapping ϕ of S into itself, how can we construct a topology in S such that the topology is in general non-trivial and renders ϕ continuous*?

Some contributions towards a solution were made by D. Ellis (1) and Mary Powderly and Hing Tong (2). They construct certain T_1-spaces, but, for example, do not even show that for every pair S, ϕ, where S is infinite, one can always construct a non-trivial T_1-space satisfying the requirements.

We shall first prove that the answer to the problem is in the affirmative even if we require S to become a *metric* space.

THEOREM I. *If S is an infinite set and $\phi S \subset S$, one can always introduce a non-discrete† metric in S under which ϕ is continuous.*

The following result is, perhaps, more interesting.

THEOREM II. *If S has at most continuously many elements and $\phi S \subset S$, then one can embed S into the discontinuum of Cantor (and so certainly into the unit segment) so that ϕ becomes continuous.*

THEOREM I', II'. *If ϕ is one-to-one (into or onto S), then the ϕ in Theorems I and II can be made to be a topological map.*

Apart from these positive results, there are some negative ones. Example I shows that, in general, *it is impossible to render (the infinite) S dense in itself so that ϕ becomes continuous.* Can the infinite S be made compact metric? Then it is necessary that S have either a countable number of elements or continuously many elements. Nevertheless we shall show in Examples II and III that this is, even under this assumption, in general impossible. In our examples ϕ is even one-to-one. The reader may verify that it is not always possible to make S connected (and ϕ continuous). However, if ϕ is one-to-one *'onto'*, the following can be proved (3):

If ϕ is one-to-one with $\phi S = S$, and S has continuously many

† i.e. there is at least one accumulation point.

elements, then S can be made compact metric and ϕ topological for every ϕ, if and only if the continuum hypothesis holds.

2. Preliminaries and proof of Theorem I

We consider the set S together with a mapping ϕ such that $\phi S \subset S$. By $\Gamma(x)$, *the orbit of* x, we shall denote the set

$$\bigcup_{k=0}^{\infty} \{\phi^k(x)\},$$

$\phi^0(x)$ denoting x. We shall call two points x, y *equivalent* if there exist non-negative integers m, n such that $\phi^m(x) = \phi^n(y)$. This is indeed an equivalence-relation as can easily be seen. The classes of equivalent points of S we shall call *ramifications*. A non-empty subset of S which consists of the orbit of any of its points is called a *cycle*. The *order* of a cycle is the number of its points. A ramification contains at most one cycle.

Proof of Theorem I. (A) Suppose that there exists at least one infinite orbit $\Gamma(x)$. Then we give the points $\phi^k(x)$ distances to each other as if they were the points $(\cos 2\pi k/t, \sin 2\pi k/t)$ in the euclidean plane, t being an irrational number, and make all other distances equal to 2. Then S is evidently metricized. Converging sequences can from a certain element onwards consist only of points of the described circle, which by the mapping ϕ only undergoes a rotation. Thus ϕ is continuous.

(B) Every orbit is finite. We distinguish between the following two cases:

(i) there exists at least one infinite ramification R,

(ii) all ramifications are finite.

(i) If there exists a point x in R such that

$$\phi^{-1}(x) = \{y \mid \phi(y) = x\}$$

is infinite, we get a non-trivial metric topology of S by introducing such a metric topology in $\phi^{-1}(x)$ with distances smaller than 1 and making all other distances equal to 1. In this topology ϕ is continuous. If $\phi^{-1}(y)$ is finite for all y in R, we proceed as follows. Since every orbit of R is finite, there exists a point x in R and a minimal positive integer k such that $\phi^k(x) = x$. The points $\phi^n(x)$ form a cycle with order k. Let x be a point of the cycle such that there exists a point a_1 in S with $\phi(a_1) = x$ and a_1 not being an element of the cycle, while

$$\{y \mid \exists\, n\, [\phi^n(y) = a_1]\}$$

is infinite, which is possible considering our initial hypotheses. Using a well-known procedure we can select a sequence of distinct points a_i $(i = 1, 2,...)$ such that $\phi(a_{i+1}) = a_i$. Now we introduce in R the following distances:

$$\rho(a_{nk}, x) = n^{-1} \quad (n = 1, 2,...),$$

$$\rho(a_{nk-i}, \phi^i(x)) = n^{-1} \quad (n = 1, 2,...; i = 1, 2,..., k-1),$$

and extend this metric linearly within the k sets $\{a_{nk-i}\}_n$. All other distances in R and S we make 1. This metric renders ϕ continuous.

(ii) There exist infinitely many finite ramifications. Each finite ramification contains one cycle. If infinitely many cycles C_i $(i = 1, 2,...)$ have the same order m, we can embed them in the euclidean plane by picking an element $c_i \in C_i$ and $\phi^n(c_i)$ giving the coordinates

$$\phi^n(c_1) = \left(\cos\frac{2\pi n}{m}, \sin\frac{2\pi n}{m}\right) \quad (n = 1, 2,..., m),$$

$$\phi^n(c_i) = \left((1-i^{-1})\cos\frac{2\pi n}{m}, (1-i^{-1})\sin\frac{2\pi n}{m}\right)$$

$$(i = 2, 3,...; n = 1, 2,..., m).$$

Making all other distances 2 we get a metric as required.

In the other case there exist cycles D_i $(i = 1, 2,...)$, the order of D_i being m_i $(m_1 > 1)$, while $m_i < m_{i+1}$. Pick an element d_i out of D_i $(i = 1, 2,...)$. Let k be a non-negative integer smaller than m_1. We embed the points

$$\phi^k(d_1),\ \phi^{sm_1+k}(d_{sm_1+k+i}) \quad (s = 0, 1,...; i = 2, 3,...)$$

in a Hilbert space by $\phi^k(d_1)$ identifying with the origin and $\phi^{sm_1+k}(d_{sm_1+k+i})$ giving the coordinates

$$x_{s+1} = i^{-1},$$

$$x_j = 0 \quad (j \neq s+1).$$

Doing this for $k = 0, 1,..., m_1-1$ in separate Hilbert spaces and making all other distances equal to 1 we get a metric which suffices.

3. Proof of Theorem II

By the *Cantor set* of a closed interval $[a, b]$ we mean the set of points $a \leqslant p \leqslant b$ such that in a triadic expansion of $(b-a)^{-1}(p-a)$ the digital 1 can be avoided.

For every fixed k we shall embed the points belonging to ramifications which contain a cycle of order k in a Hilbert space H_k. Pick out of every such cycle one point, and map the obtained set one-to-one in the

Cantor set of $[\frac{1}{2}, 1]$ on the x_1-axis. If u is the x_1-coordinate of a point p obtained in such a way, $\phi^j(p)$ will be the point

$$x_1 = u\cos\frac{2\pi l}{k}, \qquad x_2 = u\sin\frac{2\pi l}{k}, \qquad x_i = 0 \quad (i > 2).$$

We now proceed by induction, by using the induction hypothesis that the points p, with $\phi^{i-2}(p)$ in a cycle and $\phi^{i-3}(p)$ not in a cycle, are mapped in H_k such that only their first i coordinates are unequal to zero. If the point $p = (p_1, p_2, \ldots, p_i, 0, 0, \ldots)$ $(i \geqslant 2)$ is already defined, we put the set $\phi^{-1}(p)$ in a one-to-one manner in H_k such that for the coordinates of its points we have

$$x_m = p_m \quad (m \leqslant i), \qquad x_m = 0 \quad (m > i+1)$$

and x_{i+1} is a number of the Cantor set of $[\frac{1}{2}(i+1)^{-1}, (i+1)^{-1}]$ (if $i = 2$ we exclude the point already mapped).

The acquired set in H_k is zero-dimensional. Within H_k the map ϕ is certainly continuous since converging sequences are either projected or rotated under ϕ.

The points belonging to the ramifications which do not contain a cycle are mapped in a Hilbert space H_∞ in the following way. Out of every such ramification we pick an orbit (which is necessarily infinite). The set of initial points of these orbits we map in the Cantor set of $[\frac{1}{2}, 1]$ on the x_1-axis of H_∞. If u is the x_1-coordinate of such a point p, $\phi^j(p)$ will be the point

$$x_1 = u\cos\frac{2\pi l}{t}, \qquad x_2 = u\sin\frac{2\pi l}{t}, \qquad x_i = 0 \quad (i > 2),$$

t being a certain irrational number. Now exactly the same procedure can be followed as with regard to the embedding in H_k.

The diameters of the constructed sets in the various Hilbert spaces do not exceed a suitable positive number r. If two points belong to different Hilbert spaces, we give them the distance r.

In such a way we have made S into a metric space which is the union of countably many separable zero-dimensional spaces, closed in their union. Accordingly S itself is separable and zero-dimensional and thus homeomorphic to a subset of the discontinuum of Cantor.

4. Special cases: Examples

If ϕ is one-to-one, only slight modifications in the metrization procedures are necessary to make ϕ to be a topological map.

It is not always possible to make S metric and dense in itself so that ϕ becomes continuous. This is shown by the following example.

Example I. Let S be countable and ϕ be such that the ramification system of S consists of infinitely many cycles of order 1 and one cycle of order 2 (i.e. all elements of S are mapped on themselves, except one pair, the elements of which are interchanged).

An element of the cycle of order 2 cannot be the accumulation point of an infinite number of points of S.

If the potency of S is $\aleph_0$ or $\aleph$, it is not always possible to make S into a compactum, rendering ϕ continuous, as the following examples show.

Example II. Let S be countably infinite and ϕ be one-to-one onto S, so that there is only one ramification (a *strand*).

S cannot be made into a compactum: for ϕ would be a topological map, and, since at least one point would be an accumulation point, all points would be accumulation points, which is impossible in a countable compactum.

Example III. Let the ramification system of S consist of continuously many infinite disjoint orbits of certain points.

S cannot be made to a compactum: for then

$$\bigcap_{k=0}^{\infty} \phi^k(S)$$

would be non-empty as the intersection of infinitely many non-empty compact sets, but as regards the given ramification system this intersection is empty.

Remark. All embeddings of S that we have actually constructed in this paper render ϕ not only continuous but even uniformly continuous.

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A TOTALLY UNSTABLE DENUMERABLE MARKOV PROCESS

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1. Introduction

LET a temporally-homogeneous denumerable Markov process $\mathscr{P}$ be specified as usual by the set $\{p_{ij}(t): i, j \in S; t \geqslant 0\}$ of transition probabilities, where S is a denumerable set and the following relations hold:

$$\left.\begin{aligned} p_{ij}(t) &\geqslant 0 \\ \sum_{\alpha \in S} p_{i\alpha}(t) &= 1 \\ p_{ij}(u+v) &= \sum_{\alpha \in S} p_{i\alpha}(u) p_{\alpha j}(v) \\ \lim_{t \to 0} p_{ij}(t) &= p_{ij}(0) = \delta_{ij} \end{aligned}\right\}. \tag{1.1}$$

From the work of Doob (**2**) and Kolmogorov (**7**) it is known that the limits

$$q_{ij} \equiv \lim_{t \to 0} \frac{p_{ij}(t) - \delta_{ij}}{t} \tag{1.2}$$

exist, and that they are finite except perhaps when $i = j$. When $i = j$, it is usual to write $q_i \equiv -q_{ii}$ and to say that the state i is *stable* when $q_i < \infty$ and that it is *instantaneous* (or *unstable*) when $q_i = \infty$. [The terminology is that of Lévy (**8**).] At first intuition suggested that an instantaneous state must always be accompanied by an infinity of stable ones, but this is now known to be incorrect; Dobrušin (**1**) (who refers to unpublished work by Yuškevič) and Feller and McKean (**3**) have independently shown that it is possible for *all* states to be instantaneous. Such a process might be called *totally unstable*. (It is to be noted that in passing to a probabilistic description in terms of the sample functions it will be necessary to add further 'fictitious' states, but these do not participate in the analytical specification (1.1).)

The totally unstable process of Feller and McKean was constructed by first making up a certain diffusion process with the whole interval $[0, 1]$ as state-space and then showing that the system would spend virtually all its time in the states carrying rational numbers as labels. By lumping together all the states carrying irrational numbers as labels and treating them as a single 'fictitious' state '∞', it became possible to

relabel the states as $\{0, 1, 2, \ldots, \infty\}$ and all the states except '∞' were then shown to be instantaneous.

Dobrušin's method for constructing a totally unstable system is simpler in character but is still not very easy to verify; he makes up a sequence $\mathscr{P}^{(1)}, \mathscr{P}^{(2)}, \mathscr{P}^{(3)}, \ldots$ of processes each having a finite number of states in such a way that the transition probabilities for $\mathscr{P}^{(n)}$ converge (when $n \to \infty$) to transition probabilities for a process $\mathscr{P}$ which is then shown to be totally unstable.

The method to be adopted in the present paper is entirely different and has certain advantages: for example, it is possible to write down expressions in closed form for the Laplace transforms of the functions $p_{ij}(\cdot)$, and this greatly facilitates further study of the properties of the system. In brief our procedure will be to write down a linear operator Ω acting on a dense subset of the Banach space l, to verify that it satisfies the Hille–Yosida conditions, and thus to conclude that it is the infinitesimal generator of a transition semigroup; a simple calculation then reveals that the associated denumerable Markov process is totally unstable. [See also the note added in proof on p. 159.]

2. The infinitesimal generator

For each non-negative integer n we write I_n for the set of all symbols $[0, i_1, i_2, \ldots, i_n]$ (where the i's are positive integers) so that, for example, $[0, 3, 7, 18, 3] \in I_4$, and I_0 consists of the single element $[0]$. Where convenient we shall drop the square brackets and commas and write 0 for $[0]$, $0i_1 i_2$ for $[0, i_1, i_2]$, and so on. We write I for the union of all the I_n ($n = 0, 1, 2, \ldots$) and α, β, γ for typical elements of I. If $\alpha \in I_n$, then we shall write $n = \Delta(\alpha)$. The elements of the denumerable set I will be used both to label the states of the system whose motion is to be described and to label the components of the vectors $x \equiv (\ldots, x_\alpha, \ldots)$ constituting the real Banach space l, so that

$$\|x\| = \sum_{\alpha \in I} |x_\alpha| < \infty \tag{2.1}$$

for all $x = l$. It will be noticed that the elements α of I could also be used to label the vertices of an infinitely ramified tree with $[0]$ as parent vertex, and one can conveniently think of the stochastic motion as taking place on such a tree.

If $\alpha = 0i_1 i_2 \ldots i_n$ (where $n \geqslant 1$), then we define the *immediate predecessor* of α to be

$$\pi\alpha \equiv 0i_1 i_2 \ldots i_{n-1} \in I_{n-1}; \tag{2.2}$$

thus $\pi\alpha = 0$ whenever $\alpha \in I_1$, and $\pi 0$ is not defined. If $\alpha = 0i_1 i_2 \ldots i_n$

(where $n \geqslant 0$), then we denote the *immediate successors* of α by the symbols

$$\alpha j \equiv 0 i_1 i_2 \dots i_n j \in I_{n+1}, \tag{2.3}$$

where j is a positive integer. Thus

$$\pi(\alpha j) = \alpha, \tag{2.4}$$

and in general $\pi(\alpha j) \neq (\pi\alpha) j$. It follows that we can write

$$\sum_{\pi^m \beta = \alpha} \phi(\beta)$$

instead of

$$\sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \dots \sum_{j_m=1}^{\infty} \phi(\alpha j_1 j_2 \dots j_m),$$

and this contracted notation will frequently be useful.

The set I will be partly ordered by a relation '$\leqslant$', where $\alpha \leqslant \beta$ shall mean that $\alpha j_1 j_2 \dots j_m = \beta$ for some $m \geqslant 0$ and suitable positive j's; we shall then have $\pi^m \beta = \alpha$. As usual we write $\alpha < \beta$ when $\alpha \leqslant \beta$ and equality is excluded, and then we say that α '*precedes*' β.

For any $x \in l$ we write

$$x^\alpha \equiv \sum_{\alpha \leqslant \beta} x_\beta \quad (\alpha \in I); \tag{2.5}$$

such summations over the 'subtree' having α as parent vertex necessarily converge absolutely.

The specification of the system will involve an I-indexed family of real numbers $\{a_\alpha : \alpha \in I\}$ having the properties

$$\left.\begin{aligned} &\text{(i)} \quad a_0 = 0 \\ &\text{(ii)} \quad 0 < a_\alpha < \infty \quad \text{when } \alpha \neq 0 \\ &\text{(iii)} \quad S_m(\alpha) \equiv \sum_{\pi^m \gamma = \alpha} \frac{1}{c_\gamma} < \infty \quad (m \geqslant 1) \\ &\text{(iv)} \quad S_m(\alpha) \to 0 \quad \text{when } m \to \infty \end{aligned}\right\}. \tag{2.6}$$

Here c_γ is to be defined in terms of the a's by

$$c_\gamma \equiv \sum_{\beta \leqslant \gamma} a_\beta \quad (\gamma \in I); \tag{2.7}$$

it will be noticed that $c_\gamma > 0$ unless $\gamma = 0$ and that we shall never have occasion to divide by c_γ unless $\gamma \neq 0$. It should further be observed that

$$0 < S_m(\alpha) \leqslant S_{m+r}(0) \tag{2.8}$$

when $r \equiv \Delta(\alpha)$; thus in verifying (iii) and (iv) of (2.6) we need only examine what happens when $\alpha = 0$.

It must be shown by means of an example that such sets $\{a_\alpha: \alpha \in I\}$ actually exist, and the reader will be able to verify that

$$\left.\begin{aligned} a_0 &\equiv 0 \\ a_{0j_1j_2\ldots j_n} &\equiv 3^{j_1+j_2+\ldots+j_{n-1}}(3^{j_n}-1) \end{aligned}\right\}$$

is such a set; in fact

$$c_{0j_1j_2\ldots j_m} = 3^{j_1+j_2+\ldots+j_m}-1$$

and

$$S_m(0) < \frac{1}{2^{m-1}} \to 0 \quad \text{when } m \to \infty.$$

We are now in a position to introduce the operator Ω which is to generate the transition semigroup. *The domain $\mathscr{D}(\Omega)$ is to consist of all the vectors x such that*

$$\left.\begin{aligned} &\text{(i)} \quad x \in l \\ &\text{(ii)} \quad \textit{the sum } \sum_{\pi^m\beta=\alpha} |c_\beta x^\beta - x_{\pi\beta}| \textit{ is finite for each } \alpha \in I \\ &\qquad\qquad \textit{and } m \geqslant 1, \textit{ and it tends to zero when } m \to \infty \\ &\text{(iii)} \quad \sum_{\alpha\neq 0}\Big|\sum_{\pi\beta=\alpha}(c_\beta x^\beta - x_\alpha) - (c_\alpha x^\alpha - x_{\pi\alpha})\Big| < \infty \end{aligned}\right\}. \tag{2.9}$$

It will be noticed that in verifying (ii) of (2.9) we need consider only what happens when $\alpha = 0$; also (i), (ii), and (iii) must be verified in the stated order, for each implies the meaningfulness of its successor. *For each $x \in \mathscr{D}(\Omega)$ we now define Ωx by*

$$\left.\begin{aligned} (\Omega x)_0 &\equiv \sum_{\pi\beta=0}(c_\beta x^\beta - x_0) \\ (\Omega x)_\alpha &\equiv \sum_{\pi\beta=\alpha}(c_\beta x^\beta - x_\alpha) - (c_\alpha x^\alpha - x_{\pi\alpha}) \end{aligned}\right\}, \tag{2.10}$$

where in the second definition $\alpha \neq 0$. It will be observed that $\mathscr{D}(\Omega)$ is linear, that the series occurring in the definitions of $(\Omega x)_0$ and $(\Omega x)_\alpha$ $(\alpha \neq 0)$ are all absolutely convergent, and that $\sum |(\Omega x)_\alpha| < \infty$; thus Ω maps $\mathscr{D}(\Omega)$ linearly into l. We shall therefore have in Ω the infinitesimal generator of a transition semigroup associated with an (honest!) Markov process if we can show that the following (Hille–Yosida) conditions hold [cf. (4)]:

HY 1. *$\mathscr{D}(\Omega)$ is dense in l.*

HY 2. $\sum_{\alpha\in I}(\Omega x)_\alpha = 0$ *when* $0 \leqslant x \in \mathscr{D}(\Omega)$.

HY 3. *Whenever $\lambda > 0$ and $0 \leqslant x \in l$, then the equation*

$$\lambda\xi - \Omega\xi = x \quad (\xi \in \mathscr{D}(\Omega)) \tag{2.11}$$

has a unique solution ξ, and this $\xi \geqslant 0$.

(Here we write $x \geqslant 0$ to mean that $x_\alpha \geqslant 0$ for all $\alpha \in I$.)

In a first draft of this paper HY 1 was verified directly, but I am indebted to Mr. G. E. H. Reuter for pointing out to me (among other valuable suggestions) a rearrangement of the argument by which the reader will be spared a somewhat formidable computation. We shall begin by verifying HY 2 and HY 3, and it will then follow that (2.11) has a unique solution for *all* (and not merely for positive) x in l; we write $\xi \equiv J_\lambda x$ for this solution and we recall that $J_\lambda x$ is positive whenever x is so; in fact λJ_λ will be (for each positive λ) a transition operator having $\mathscr{D}(\Omega)$ as its exact range. It will then at once follow that $\mathscr{D}(\Omega)$ is dense (so that HY 1 is satisfied) if it can be shown that

$$\lim_{\lambda\to\infty} \|\lambda J_\lambda x - x\| = 0$$

for some fundamental set of vectors x in l. Such a set is $\{u(\alpha): \alpha \in I\}$, where

$$(u(\alpha))_\beta \equiv \delta_{\alpha\beta} \quad (\beta \in I), \tag{2.12}$$

and, if (as will otherwise be convenient) we write

$$\phi_{\alpha\beta}(\lambda) \equiv (J_\lambda u(\alpha))_\beta, \tag{2.13}$$

then

$$\|\lambda J_\lambda u(\alpha) - u(\alpha)\| = 2\{1-\lambda\phi_{\alpha\alpha}(\lambda)\},$$

so that once HY 2 and HY 3 have been verified we are at liberty to replace HY 1 by

$$\text{HY 1'.} \quad \lim_{\lambda\to\infty} \lambda\phi_{\alpha\alpha}(\lambda) = 1 \quad (\alpha \in I).$$

It will be noticed that $\phi_{\alpha\beta}(\lambda)$ is the βth component of ξ, where

$$\lambda\xi - \Omega\xi = u(\alpha).$$

3. The Hille–Yosida conditions are satisfied

We shall first show that $\sum (\Omega x)_\alpha = 0$ for *all* $x \in \mathscr{D}(\Omega)$ and this will establish slightly more than is required by HY 2. From the definition of Ωx we have

$$\sum_{\Delta(\alpha)\leqslant n} (\Omega x)_\alpha = \sum_{\Delta(\beta)=n+1} (c_\beta x^\beta - x_{\pi\beta}),$$

so that

$$\Big|\sum_{\Delta(\alpha)\leqslant n} (\Omega x)_\alpha\Big| \leqslant \sum_{\pi^{n+1}\beta=0} |c_\beta x^\beta - x_{\pi\beta}|,$$

and the conclusion now follows from (2.9)(ii).

A similar argument shows that, if $x \in \mathscr{D}(\Omega)$ and $\alpha \neq 0$, then

$$\sum_{\substack{\alpha\leqslant\beta \\ \Delta(\beta)\leqslant\Delta(\alpha)+n}} (\Omega x)_\beta = x_{\pi\alpha} - c_\alpha x^\alpha + \sum_{\pi^{n+1}\gamma=\alpha} (c_\gamma x^\gamma - x_{\pi\gamma}),$$

so that

$$\left.\begin{aligned} (\Omega x)^0 &= 0 \\ (\Omega x)^\alpha &= x_{\pi\alpha} - c_\alpha x^\alpha \quad (\alpha \neq 0) \end{aligned}\right\}, \tag{3.1}$$

when $x \in \mathscr{D}(\Omega)$.

As a first step in the verification of HY 3 we shall show that the equation $\Omega\xi = \lambda\xi$ has no non-null solution $\xi \in \mathscr{D}(\Omega)$ when $\lambda > 0$. For, if $\Omega\xi = \lambda\xi$, then, in virtue of (3.1), we shall have

$$\lambda\xi^0 = 0$$

and

$$\lambda\xi^\alpha = \xi_{\pi\alpha} - c_\alpha\xi^\alpha \quad (\alpha \neq 0),$$

so that

$$\xi^{\beta j} = \xi_\beta/(\lambda + c_{\beta j}),$$

$$\xi^\beta = \xi_\beta\left\{1 + \sum_{\pi\gamma=\beta} \frac{1}{\lambda + c_\gamma}\right\},$$

and

$$\xi^{\beta j} = \frac{\xi^\beta}{\lambda + c_{\beta j}}\left\{1 + \sum_{\pi\gamma=\beta} \frac{1}{\lambda + c_\gamma}\right\}^{-1}.$$

It has already been seen that $\xi^0 = 0$, and the above recurrence relation therefore shows that $\xi^\beta = 0$ for all $\beta \in I$. That $\xi \equiv 0$ now follows from the identity

$$\xi_\alpha = \xi^\alpha - \sum_{j=1}^{\infty} \xi^{\alpha j}.$$

We now know that (2.11) has *at most* one solution when $\lambda > 0$; we shall establish the *existence* of a solution (when $\lambda > 0$ and $0 \leqslant x \in l$) by actually constructing it. We shall first write down a recurrence relation determining ξ_α in terms of $\xi_{\pi\alpha}$ ($\alpha \neq 0$) and the definition of the vector ξ will then be completed by specifying the 'first term' ξ_0. It will then be shown that the vector ξ so constructed lies in $\mathscr{D}(\Omega)$, is positive, and satisfies (2.11). *The recurrence relation is*

$$\xi_\alpha\left\{1 + \sum_{\pi\beta=\alpha} \frac{1}{\lambda + c_\beta}\right\} = \frac{x_\alpha + \xi_{\pi\alpha}}{\lambda + c_\alpha} + \frac{1}{\lambda + c_\alpha} \sum_{\pi\beta=\alpha} \frac{a_\beta x^\beta}{\lambda + c_\beta}, \tag{3.2}$$

where $\alpha \neq 0$; this will determine ξ in terms of ξ_0 because each of the series occurring in (3.2) is convergent, and evidently we shall have $\xi \geqslant 0$ if (*as we shall*) we make $\xi_0 \geqslant 0$. Now (3.2) gives

$$\xi_\alpha + \sum_{\pi\beta=\alpha} \frac{x^\beta + \xi_{\pi\beta}}{\lambda + c_\beta} = \frac{x^\alpha + \xi_{\pi\alpha}}{\lambda + c_\alpha} \quad (\alpha \neq 0),$$

and iteration then gives

$$\sum_{r=0}^{n-1} \sum_{\pi^r\beta=\alpha} \xi_\beta + \sum_{\pi^n\beta=\alpha} \frac{x^\beta + \xi_{\pi\beta}}{\lambda + c_\beta} = \frac{x^\alpha + \xi_{\pi\alpha}}{\lambda + c_\alpha} \tag{3.3}$$

for each $n \geqslant 1$ and $\alpha \neq 0$.

One consequence of (3.3) is that

$$\xi^{0j} \leqslant \frac{x^{0j} + \xi_0}{\lambda + c_{0j}} \leqslant \frac{x^{0j}}{\lambda} + \frac{\xi_0}{c_{0j}},$$

so that
$$\xi^0 \leqslant \xi_0 + \frac{\|x\|}{\lambda} + \xi_0 \sum_{\pi\beta=0} \frac{1}{c_\beta} < \infty.$$

Thus $0 \leqslant \xi \in l$ if $\xi_0 \geqslant 0$. Continuing this line of argument we find that

$$0 \leqslant \sum_{\pi^n\beta=\alpha} \frac{x^\beta + \xi_{\pi\beta}}{\lambda + c_\beta} \leqslant \frac{1}{\lambda} \sum_{\alpha \leqslant \pi^n\gamma} x_\gamma + \|\xi\| \sum_{\pi^n\beta=\alpha} \frac{1}{c_\beta}$$

when $\xi_0 \geqslant 0$, and each term on the right-hand side tends to zero when $n \to \infty$. Thus (3.3) gives

$$(\lambda + c_\alpha)\xi^\alpha = x^\alpha + \xi_{\pi\alpha} \quad (\alpha \neq 0) \tag{3.4}$$

provided that $\xi_0 \geqslant 0$.

From the last result it follows that

$$\sum_{\pi^m\beta=\alpha} |c_\beta \xi^\beta - \xi_{\pi\beta}| \leqslant \sum_{\pi^m\beta=\alpha} (x^\beta + \lambda\xi^\beta) = \sum_{\alpha \leqslant \pi^m\gamma} (x_\gamma + \lambda\xi_\gamma)$$

when $\xi_0 \geqslant 0$, and the right-hand side is finite and tends to zero when $m \to \infty$. Also (if $\alpha \neq 0$ and $\xi_0 \geqslant 0$)

$$\sum_{\pi\beta=\alpha} (c_\beta \xi^\beta - \xi_\alpha) - (c_\alpha \xi^\alpha - \xi_{\pi\alpha}) = \sum_{\pi\beta=\alpha} (x^\beta - \lambda\xi^\beta) - (x^\alpha - \lambda\xi^\alpha) = \lambda\xi_\alpha - x_\alpha,$$

so that $0 \leqslant \xi \in \mathscr{D}(\Omega)$. The last relation further shows that

$$(\lambda\xi - \Omega\xi)_\alpha = x_\alpha$$

at least when $\alpha \neq 0$, if $\xi_0 \geqslant 0$, so it only remains for us to show that a non-negative value can be assigned to ξ_0 in such a way that

$$(\lambda\xi - \Omega\xi)_0 = x_0.$$

We now choose ξ_0 so that

$$\xi_0 \left\{1 + \sum_{\pi\beta=0} \frac{1}{\lambda + c_\beta}\right\} = \frac{1}{\lambda}\left\{x_0 + \sum_{\pi\beta=0} \frac{a_\beta x^\beta}{\lambda + c_\beta}\right\}. \tag{3.5}$$

From (3.4) we know that

$$\xi^0 = \xi_0 + \sum_{\pi\beta=0} \xi^\beta = \sum_{\pi\beta=0} \frac{x^\beta}{\lambda + c_\beta} + \xi_0 \left\{1 + \sum_{\pi\beta=0} \frac{1}{\lambda + c_\beta}\right\},$$

and so (because $a_\beta = c_\beta$ when $\pi\beta = 0$) we shall have

$$\xi^0 = x^0/\lambda. \tag{3.6}$$

This last relation can be written in the form

$$\lambda\xi_0 - \sum_{\pi\beta=0} (x^\beta - \lambda\xi^\beta) = x_0,$$

and in virtue of (3.4) this is

$$\lambda\xi_0 - \sum_{\pi\beta=0} (c_\beta \xi^\beta - \xi_0) = x_0,$$

as required.

Conditions HY 2 and HY 3 have thus been verified. Now from (3.2) and (3.5) we find that

$$\lambda\phi_{00}(\lambda) = \left\{1 + \sum_{\pi\beta=0} \frac{1}{\lambda+c_\beta}\right\}^{-1},$$

and that

$$\lambda\phi_{\alpha\alpha}(\lambda) = \frac{\lambda}{\lambda+c_\alpha}\{1+\phi_{\alpha,\pi\alpha}(\lambda)\}\left\{1 + \sum_{\pi\beta=\alpha} \frac{1}{\lambda+c_\beta}\right\}^{-1}$$

when $\alpha \neq 0$, and HY 1′ is then easily verified on noting that

$$0 \leqslant \phi_{\alpha,\pi\alpha}(\lambda) \leqslant (J_\lambda u(\alpha))^0 = \lambda^{-1} \to 0 \quad (\lambda \to \infty).$$

It therefore follows from the general theory [see, for example, (4)] that Ω on $\mathscr{D}(\Omega)$ is the infinitesimal generator of a transition semigroup $\mathscr{S}$ associated with an (honest!) Markov process $\mathscr{P}$, the transition probabilities $p_{ij}(t)$ for which are the unique bounded continuous solutions to

$$\phi_{\alpha\beta}(\lambda) = \int_0^\infty e^{-\lambda t} p_{\alpha\beta}(t)\, dt \quad (\lambda > 0),$$

where $\phi_{\alpha\beta}(\lambda)$ is the βth component of ξ when $x \equiv u(\alpha)$. It is a simple matter to obtain explicit formulae for the functions $\phi_{\alpha\beta}(\lambda)$ from the defining relations (3.5) and (3.2); for each pair (α, β) the resulting formula depends on the relative situation of the vertices labelled α and β in the infinitely ramified tree.

4. The Markov process $\mathscr{P}$ is totally unstable

It was proved in (4) that

$$q_\alpha = \lim_{\lambda\to\infty} \lambda\{1-\lambda\phi_{\alpha\alpha}(\lambda)\}; \tag{4.1}$$

thus for the Markov process $\mathscr{P}$ we shall have

$$q_0 = \lim_{\lambda\to\infty} \sum_{\pi\beta=0} \frac{\lambda}{\lambda+c_\beta}\left\{1 + \sum_{\pi\gamma=0} \frac{1}{\lambda+c_\gamma}\right\}^{-1}$$

and

$$q_\alpha = \lim_{\lambda\to\infty} \left\{\frac{\lambda c_\alpha}{\lambda+c_\alpha} - \frac{\lambda^2\phi_{\alpha,\pi\alpha}(\lambda)}{\lambda+c_\alpha} + \sum_{\pi\beta=\alpha} \frac{\lambda}{\lambda+c_\beta}\right\}\left\{1 + \sum_{\pi\gamma=\alpha} \frac{1}{\lambda+c_\gamma}\right\}^{-1}$$

when $\alpha \neq 0$. Now

$$\lim_{\lambda\to\infty} \sum_{\pi\gamma=\alpha} \frac{1}{\lambda+c_\gamma} = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \sum_{\pi\beta=\alpha} \frac{\lambda}{\lambda+c_\beta} = \infty$$

for all α in I; also

$$\lim_{\lambda\to\infty} \lambda\phi_{\alpha,\pi\alpha}(\lambda) = p_{\alpha,\pi\alpha}(0+) = 0,$$

so that $$q_\alpha = \infty \quad (\text{all } \alpha \in I) \tag{4.2}$$

and $\mathscr{P}$ *is totally unstable.*

5. Some further results

The analytical properties of the process $\mathscr{P}$ and the associated semi-group $\mathscr{G}$ have been examined in considerable detail and some of the more interesting results which have emerged from this examination will be recorded here; the proofs will be omitted in the interests of brevity.

(1) The non-diagonal elements of the q-matrix are as follows:

$$\left.\begin{aligned} q_{\pi\beta,\beta} &= 1 \quad (\text{all } \beta \neq 0) \\ q_{\alpha\beta} &= a_{\beta j} \quad (\beta j \leqslant \alpha) \\ q_{\alpha\beta} &= 0 \quad (\beta \not\leqslant \alpha,\ \alpha \neq \pi\beta) \end{aligned}\right\}. \tag{5.1}$$

If we say that 'β is q-accessible from α' when $q_{\alpha\beta} > 0$, then it will be seen that in the present example β will be q-accessible from α if and only if β is *either* any predecessor of α *or* any *immediate* successor of α. According to Professor Feller [in a personal communication] this fact distinguishes the present example from that of Feller and McKean (for which $q_{\alpha\beta} = 0$ whenever $\alpha \neq \beta$). In the present example

$$q_\alpha = \sum_{\beta \neq \alpha} q_{\alpha\beta} = \infty \quad (\text{all } \alpha \in I).$$

(2) When $t \to \infty$, $$p_{\alpha\beta}(t) \to P_\beta,$$

where P_β is positive, does not depend on α, and satisfies $\sum P_\beta = 1$. The following recurrence relations determine the P_β:

$$\left.\begin{aligned} P_0 &= \left\{1 + \sum_{\pi\alpha = 0} \frac{1}{c_\alpha}\right\}^{-1} \\ P_\beta &= \frac{P_{\pi\beta}}{c_\beta}\left\{1 + \sum_{\pi\gamma = \beta} \frac{1}{c_\gamma}\right\}^{-1} \quad (\beta \neq 0) \end{aligned}\right\}. \tag{5.2}$$

If we assign probability P_β to the choice of $\beta \in I$ as initial state, then $\mathscr{P}$ becomes a stationary process. It is further found [in the language of (6)] that the states form a single positive class and that the system is 'strongly ergodic'.

(3) We now adjoin 'ideal' states

$$[0, i_1, i_2, i_3, \ldots]$$

to I (where the i's are positive integers) and allow them to participate in the partial ordering of I in the obvious way. We write $\mathscr{T}$ for I when

so augmented, τ for a typical element of $\mathscr{T}$ and $\mathscr{T}(\alpha)$ for the sub-tree

$$\mathscr{T}(\alpha) \equiv \{\tau: \tau \in \mathscr{T}, \alpha \leqslant \tau\} \tag{5.3}$$

when $\alpha \in I$. We then topologize the augmented tree $\mathscr{T}$ by taking the sub-trees $\mathscr{T}(\alpha)$ and their set-theoretic complements as a system of sub-basic open sets. With this topology $\mathscr{T}$ is a totally disconnected compact Hausdorff space containing I as a dense subset. If J is the set $\mathscr{T} - I$ of adjoined states, then J is also dense in $\mathscr{T}$. Also, if

$$\theta = [0, i_1, i_2, i_3, \ldots] \in J$$

and if

$$\theta_n \equiv [0, i_1, i_2, \ldots, i_n] \in I_n,$$

then $\theta_n \to \theta$ when $n \to \infty$.

(4) We next introduce the Banach space $C(\mathscr{T})$ of all real-valued continuous functions having $\mathscr{T}$ as domain and with the customary (uniform) norm. If $\beta \in I$ and if we write

$$\zeta[\tau; \beta] \equiv \begin{cases} 1 & \text{when } \tau \in \mathscr{T}(\beta), \\ 0 & \text{when } \tau \notin \mathscr{T}(\beta), \end{cases} \tag{5.4}$$

then we obtain an element $\zeta[\,.\,; \beta]$ of this function space and the Stone–Weierstrass theorem shows that the set of all $\zeta[\,.\,; \beta]$ with $\beta \in I$ is fundamental in $C(\mathscr{T})$.

If z is an arbitrary element of $C(\mathscr{T})$, then we can associate with it the element of m (the Banach space of all bounded real-valued I-indexed sequences) obtained when the domain of the function z is restricted from $\mathscr{T}$ to I. Not all the vectors in m can be obtained in this way, however. Let the linear subset of m so obtained be denoted by Z. The natural mapping from $C(\mathscr{T})$ to Z is a (1:1) isometric linear isomorphism, so that Z is a closed subspace of m equivalent to $C(\mathscr{T})$, and in fact Z coincides with the closed subspace of m spanned by the *vectors* $\zeta(\beta)$ obtained by the restriction of the domain of the *functions* $\zeta[\,.\,; \beta]$, so that

$$(\zeta(\beta))_\alpha = \begin{cases} 1 & \text{if } \beta \leqslant \alpha, \\ 0 & \text{if } \beta \not\leqslant \alpha. \end{cases}$$

With the usual notation (y, x) for the scalar product of $y \in m$ and $x \in l$ we shall have

$$(\zeta(\beta), x) = x^\beta.$$

Also, if $z \in C(\mathscr{T})$ and if $\theta \in J$, then the relation

$$z[\theta] = \lim_{n \to \infty} z[\theta_n]$$

enables one to calculate the values of the *function* $z \in C(\mathscr{T})$ from the components of the associated *vector* $z \in Z$: in fact

$$z[\tau] = z_\tau \qquad \text{when } \tau \in I,$$
$$z[\tau] = \lim_{n\to\infty} z_{\tau_n} \qquad \text{when } \tau \in J.$$

(5) The theory of the adjoint semigroup $\mathscr{G}^\oplus$ in the sense of Phillips (9) can now be applied; this has been summarized from a probabilistic standpoint in (5) and the notations and terminology of that paper will be employed here. It turns out that the semigroup $\mathscr{G}^\oplus$ adjoint in the sense of Phillips to the semigroup $\mathscr{G}$ of §§ 2 and 3 acts on the subspace $X^\oplus \equiv Z$ of m; the resolvent operator J_λ is strongly compact, and so $\mathscr{G}$ is a reflexive semigroup symmetrically related to $\mathscr{G}^\oplus$. Because Z is equivalent to $C(\mathscr{T})$, the dual semigroup $\mathscr{G}^\oplus$ can be transferred so as to act on this space of continuous functions, and interesting questions now arise concerning the relation between the ideal states added to I to yield the compactified state-space $\mathscr{T}$ (the carrier-space of the continuous functions on which the dual semigroup $\mathscr{G}^\oplus$ acts) and the ideal states which would have to be adjoined to I to permit an adequate description of the stochastic motion. It is hoped to give further study to such questions in a wider context and accordingly the preliminary results obtained in connexion with the present example will not be further reported here. We therefore conclude with a specification of the infinitesimal generator $\Omega^\oplus$ of the dual semigroup $\mathscr{G}^\oplus$. This operator acts on the set of all m-vectors y which satisfy the conditions

(i) $$\left|\sum_{0<\beta\leqslant\alpha} c_\beta(y_{\pi\beta}-y_\beta)\right| \leqslant k(y) < \infty \tag{5.5}$$

for all $\alpha \neq 0$;

(ii) the vector $y\Omega^\oplus$ defined by

$$(y\Omega^\oplus)_\alpha \equiv \sum_{0<\beta\leqslant\alpha} c_\beta(y_{\pi\beta}-y_\beta) - \sum_{\pi\beta=\alpha}(y_\alpha - y_\beta) \tag{5.6}$$

lies in the subspace Z.

When y is an m-vector satisfying (i) and (ii), i.e. when $y \in \mathscr{D}(\Omega^\oplus)$, then $y\Omega^\oplus$ is given by (5.6).

Note added in proof. Professor David Blackwell informs me that he has recently constructed yet another example of a totally unstable process. His method is extremely simple and direct; no use is made of semigroup techniques, and indeed it might be quite a difficult task to calculate the infinitesimal generator Ω of the Blackwell process.

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